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Abstract: The subject of the paper is the analysis of sensitivity of a thin elastic spherical shell to the change of its shape associated with forming a small circular opening, far from the loading applied. The analysis concerns the elastic potential of the shell. The sensitivity of this functional is measured as a topological derivative, introduced for the plane elasticity problem by Sokolowski and Żochowski (1997) and extended here to the case of a spherical shell. A proof is given that : i) the first derivative of the functional with respect to the radius of the opening vanishes, and : ii) the second derivative does not blow up. A partially constructive formula for the second derivative or for the topological derivative is put forward. The theoretical considerations are confirmed by the analysis of a special case of a shell loaded rotationally symmetric, weakened by an opening at its north-pole. The whole treatment is based on the Niordson-Koiter theory of spherical shells, belonging to the family of correct first order shell models of Love.

Key-words: shape optimization, shape derivative, topological derivative, asymptotic expansion, inverse problem

(Résumé : tsvp)

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Coques optimales formés sur une sphère. La méthode de la dérivée topologique.

Résumé : Le sujet de cet article est l'analyse de la sensibilité d'une coque mince, élastique et sphérique vis-à-vis de la modification de sa forme causée par la création d'une petite ouverture circulaire, loin de la charge appliquée. L'analyse concerne le potentiel élastique de la coque. La sensibilité de cette fonctionnelle est mesurée comme une dérivée topologique, notion introduite pour le problème de l'élasticité plane par Sokolowski et Zochowski (1997) et étendu ici au cas d'une coque sphérique. On démontre les résultats suivantes: i) la dérivée première de la fonctionnelle par rapport au rayon de l'ouverture s'annule, et : ii) la dérivée seconde n'explose pas.

On propose une formule partiellement constructive pour la dérivée seconde ou la dérivée topologique. Les considérations théorique sont confirmées par l'analyse d'un cas spécial de coque chargée suivant une symétrie de révolution et fragilisée par une ouverture à son pôle nord. Tout ce travail est basé sur la théorie de Niordson-Koiter des coques sphérique, appartenant à la famille des modèles de coques exactes du premier ordre de Love.

Mots-clé : optimisation de formes, dérivée topologique, coque mince

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Notation used in the paper

1. Spaces

$C^s(\mathcal{O})$: the space of functions given on $\mathcal{O} \subset \mathbb{R}^2$, continuous up to their s th derivatives ; $\mathcal{O}$ is a domain, open and bounded

$H^m(\mathcal{O})$, $H_0^m(\mathcal{O})$: Sobolev spaces

V_0 : kinematically admissible space for a clamped shell, see (2.26)

V_1 : kinematically admissible space for a partly clamped shell, see (2.31)

V_{θ_0} : see Sec. 4.2

$\mathcal{R}$, $\widetilde{\mathcal{R}}$: the classes of rigid motions of the Koiter and Koiter-Niordson shells, respectively

2. Sets and domains

$\mathcal{O}$: as above

S : middle surface of a shell

$\Gamma = \partial S$: the boundary line of S, parametrized by s ; Γ_σ : the loaded part ; Γ_u : the clamped part

3. Parametrizations

$\{x^i\}$, $i = 1, 2, 3$: Cartesian coordinate system

(ξ^1, ξ^2) : curvilinear coordinates on S

$\xi = (\xi^1, \xi^2)$: point on S

$\mathbf{\Phi} : \mathcal{O} \to S$: the mapping identified with $x^i(\xi)\mathbf{e}_i$

$\{\mathbf{e}_i\}$: Cartesian basis of $\{x^i\}$

$\{\mathbf{a}_i\}$: basis on S

$\xi^1 = \theta$, $\xi^2 = \phi$: spherical coordinates : meridional and circumferential, respectively

$\theta = 0$: the north-pole of a sphere

$\theta \in [0, \pi/2]$, $\phi \in [0, 2\pi]$: a hemisphere

4. Operations, conventions

The small Greek indices $\alpha, \beta, \lambda, \mu, \sigma, \gamma \dots$ run over $\{1, 2\}$; the summation convention is used for indices at different levels.

$\{\delta_{ij}\}$: Kronecker delta

$\mathbf{a} \times \mathbf{b}$: vector product of $\mathbf{a}$ and $\mathbf{b}$ (vectors)

$\mathbf{a} \otimes \mathbf{b}$: tensor product of $\mathbf{a}$ and $\mathbf{b}$

$(\ )_{,\alpha} = \partial / \partial \xi^\alpha$

$(\ )|_\alpha$: covariant derivative on S, see (2.8) ; for a scalar f we have $f|_\alpha = f_{,\alpha}$

Δw : the Laplace-Beltrami operator on S, see (2.9) ; w is a scalar here

T^*_α, $T^*_{\alpha\beta}$: physical components of (T_α) and $(T_{\alpha\beta})$

$\mathbf{n} = n^\alpha \mathbf{a}_\alpha$: vector normal to Γ and tangent to S

$\mathbf{t} = t^\alpha \mathbf{a}_\alpha$: vector tangent to Γ and S

$\dfrac{\partial f}{\partial n} = f|_\alpha n^\alpha = f_{,\alpha} n^\alpha$

$\dfrac{\partial f}{\partial s} = f|_\alpha t^\alpha$: directional derivatives with respect to vectors $\mathbf{n}$ and $\mathbf{t}$, respectively

$(\ )' = d/d\theta_0$

5. Geometrical objects on S

$(a_{\alpha\beta})$: covariant components of the metric tensor on S

$(a^{\alpha\beta})$: contravariant components of this tensor

$a = \det(a_{\alpha\beta})$; $\epsilon^{\alpha\beta}$, $\epsilon_{\alpha\beta}$: components of Ricci tensor

$(b_{\alpha\beta})$: curvature tensor of S

$\left(\Gamma^\alpha_{\beta\gamma}\right)$: Christoffel symbols on S

$(R_{\alpha\beta\gamma\sigma})$: the mixed curvature tensor of S

6. Deformation measures and displacements

$\mathbf{u} = (u_1, u_2, w)$: displacement vector of points of S, with components referred to $(\mathbf{a}^i)$

$\mathbf{u} = (u, v, w)$: physical components of this vector

$\mathbf{v} = (v_1, v_2, v)$: vector of trial displacements of points of S, with components referred to the basis $(\mathbf{a}^i)$

$\mathbf{v} = (\overline{u}, \overline{v}, \overline{w})$: physical components of $\mathbf{v}$

$u_\alpha^{(1)}$, $w^{(1)}$: displacements associated with isometric deformations, see (3.37)

$\mathbf{u}_{\theta_0} = (u_{\theta_0}, v_{\theta_0}, w_{\theta_0})$: displacements of the shell of Fig. 3.3 and Fig. 4.2

$\check{w}_{\theta_0}$: non-dimensional deflection (w_{θ_0})

$u_n = \mathbf{u}.\mathbf{n}$, $v_n = \mathbf{v}.\mathbf{n}$: displacements along $\mathbf{n}$ of Γ

$u_t = \mathbf{u}.\mathbf{t}$, $v_t = \mathbf{v}.\mathbf{t}$: displacements along $\mathbf{t}$ of Γ

$\gamma_{\alpha\beta}(\mathbf{u})$: membrane deformations (2.20)

$\varepsilon_{\alpha\beta}(\mathbf{u})$: its physical components (3.47)

$\rho_{\alpha\beta}(\mathbf{u})$: Koiter's changes of curvature, (2.21)

$\tilde{\rho}_{\alpha\beta}(w)$: Niordson-Koiter changes of curvature of a spherical shell

$\kappa_{\alpha\beta}(w)$: physical components of $\tilde{\rho}_{\alpha\beta}(w)$

$\mathbf{v}_0$: translation vector

Ω_0 : vector of infinitesimal rotation

7. Loading, stress and couple resultants

$(\mathcal{N}^{\alpha\beta})$: membrane forces or stress resultants within Koiter's theory

$(\widetilde{N}^{\alpha\beta})$: membrane forces or stress resultants in a spherical shell ; Koiter-Niordson theory

$(M^{\alpha\beta})$: bending and torsional moments or couple resultants. The same notation is used for physical components

$(N^{\alpha\beta})$: physical components of $(\widetilde{N}^{\alpha\beta})$

$(N^{\alpha\beta}_{\theta_0})$, $(M^{\alpha\beta}_{\theta_0})$: quantities referring to a shell with opening $r = R\sin\theta_0$, see Fig. 4.2

(T^{α}) : the effective membrane edge forces at γ (Koiter's theory) see (2.38)

Q : the effective tansverse force at Γ, see (2.38)

M_B : the effective bending moment at Γ, (2.38)

$N = \widetilde{N}^{\alpha\beta} n_\alpha n_\beta$: the effective membrane edge force along $\mathbf{n}$

$S = \widetilde{N}^{\alpha\beta} n_\alpha t_\beta$: the effective membrane edge force along $\mathbf{t}$
(both : Koiter-Niordson theory of spherical shells)

$(\widehat{T}^\alpha)$: the applied loading tangent to S along Γ, referred to $\mathbf{a}_\alpha$

$\widehat{N} = \widehat{T}^\alpha n_\alpha$: its component along $\mathbf{n}$

$\widehat{S} = \widehat{T}^\alpha t_\alpha$: its component along $\mathbf{t}$

$\widehat{Q}$: the applied loading transverse to S along Γ

$\widehat{M}_B$: bending moment applied to Γ

$\widehat{M}_B > 0$: if it causes $\dfrac{\partial \mathbf{w}}{\partial n} > 0$

$\mathbf{p} = (p^i)$: density of surface loading

8. Material constants and tensors of elastic moduli

E : Young's modulus

ν : Poisson's ratio

h : thickness of the shell

R : radius of a spherical shell

h/R : ratio of thinness

$(C^{\alpha\beta\lambda\mu})$: tensor of the reduced elastic moduli according to the plane-stress assumption

$(A^{\alpha\beta\lambda\mu})$: membrane stiffnesses, (2.23)

$(D^{\alpha\beta\lambda\mu})$: bending stiffnesses, (2.23)

ρ, $\tilde{\rho}$: auxiliary parameters, see (3.40)

ζ, ω : see (3.103) ; complex parameters

α, β : see (3.103)

$\sigma = a + ib$: a complex number, def. by (3.61)

$k = \dfrac{h^2}{12R^2}$: see (3.32)

9. Functionals, bilinear forms

$a_1(.,.)$: the bilinear form of the Koiter shell model

$a_2(.,.)$: the bilinear form of the Koiter-Niordson model of spherical shells

$f_1(.)$: the linear form associated with surfacial loading

$f_2(.)$: the linear form associated with edge loading

$J(\mathbf{u})$: potential energy

$\check{J}(\mathbf{u})$: non-dimensional counterpart of $J(\mathbf{u})$

$j(\theta_0)$: see (4.7)

$a_{\theta_0}(.,.)$: the bilinear form for a spherical shell of Fig. 3.3

10. Functions

$P_\sigma^m(x)$: associated Legendre function of degree σ and order m

$R_\sigma^m(\theta)$, $S_\sigma^m(\theta)$: real and imaginary parts of $P_\sigma^{-m}(\cos\theta)$, see (3.66)

Ξ : auxiliary function given by (3.36)

Φ : stress function, see (3.28)

$\widetilde{\Phi} = EhR\Phi$

$h_2(\theta_0)$, $h_6(\theta_0)$

$g_2(\theta_0)$, $g_6(\theta_0)$: see (3.102)

$a_{N_i}(\theta)$, $a_{M_i}(\theta)$, $a_{Q_i}(\theta)$: see (3.79)-(3.86)

$\tilde{a}_{M_i}(\theta)$, $\tilde{a}_{Q_i}(\theta)$: see (5.5)

$A_{i(m)}$, X_i, Z_i : integration constants

1. Introduction

A range of applicability of the classical methods of optimization of shapes of shells, with using the non-linear programming methods, is well presented in the paper by Bletzinger and Ramm (1993). The coordinates of nodes of the grid constructed on a shell are the desing variables. Within this approach the topological characteristics of the shell domain do not undergo any changes.
One can indicate two methods of further steps of optimization which change the shell domain :
i) the regularization technique of admitting the composite regions,
ii) the bubble method.
Both the methods have been applied mainly to plate optimization problems. The method (i) consists in admitting new composite regions in which the voids appear at a microscale. Its theoretical foundations are given in Kohn and Strang (1986). The optimal bending plates have been found in Gibianskii and Cherkaev (1984). The in-plane problem is discussed in Allaire and Kohn (1993), cf. the book by Bendsøe (1995).
The effective characteristics of composite domains are given by the homogenization formulae. For thin bending plates such formulae have been found by Duvaut and Metellus (1976). Their counterparts for thin Koiter's shells can be read off from Lewiński and Telega (1988). A justification of these formulae has been published only recently, cf. Telega and Lewiński (1998). These homogenization results have not up till now been used for solving the optimization problems.
The method (ii) has been originated in Eschenauer et al. (1993, 1994), cf. Schumacher (1995) and there applied to various optimization problems for plates in bending as well as subjected to the in-plane loadings. However, a crucial definition of so-called characteristic function has only been given for the minimum compliance problem, cf. Eq. (25) in Eschenauer et al. (1994). The method (ii) has not been applied to shells.
Recently Sokolowski and Żochowski (1997) have introduced the notion of a topological derivative of shape functionals. One can show with ease that in the case of the compliance minimization the topological derivative differs in a factor from the relevant characteristic function of Eschenauer et al. (1994). Indeed, in the minimum compliance problem the expression (37) from Sokolowski and

Żochowski (1997) reduces to

$$J''_\sigma(0) = -\frac{2\pi}{E}[(\sigma_{\mathrm{I}} + \sigma_{\mathrm{II}})^2 + 2(\sigma_{\mathrm{I}} - \sigma_{\mathrm{II}})^2],$$

since here $u = v$, $p = 1$ and $s_{22} = 1/E$; E represents the Young modulus, while σ_{I}, σ_{II} are principal stresses. This expression differs in the factor $(1/4\pi)$ from the function (25) of Eschenauer et al. (1994). In particular, the method of topological derivative provides a rigorous technique of finding the characteristic functions of the bubble method.

If applied to shape optimization problems the method (i) introduces voids at a microscale or in the cells of composites which are used to find the regularized solutions. This leads frequently to appearing voids in the finite element mesh, which is called a checkerboard pattern. Special methods of filtering have to be used to suppress this parasitic numerical effect. They are still under study, see Bendsøe (1995).

In the method (ii) the infinitesimal voids (or bubbles) grow to form large openings. How to change and update their geometry is also still under study.

The present paper is aimed at deriving an analytical formula for a topological derivative of the compliance functional for a thin elastic spherical shell. Its specific geometry makes it possible to simplify the Koiter shell equations (see Bernadou (1995)). The Koiter equations are recalled in Sec. 2. Their specific version, called Koiter-Niordson equations for spherical shells, is put forward in Sec. 3, mainly after Niordson (1985).

Thanks of introducing the stress function (3.29) the whole problem can be reduced to two coupled equations for two scalar functions, and then, to one equation (3.36). A complete solution is represented by (3.71)-(3.74). The form of this solution determines distribution of membrane forces and moments around the openings, especially when their dimensions are smaller and smaller. A detailed analysis of these singularities of the general solution has made it possible to note that the first derivative of the compliance functional (with respect to the opening dimension) vanishes. This remark is also justified by the study of a special case of rotational symmetry.

The expression for the second derivative of the compliance functional (also with respect to the opening dimension) is equivalent to the topological derivative. This expression is composed of two terms, see (4.29). The first term depends on the invariants of the stress and couple resultant tensors concerning the problem of a shell without opening at the point. We conjecture that the second term of (4.29) vanishes, since for the plane cases ($R \to \infty$) this holds. The proof of this conjecture is still unknown.

2. Recapitulation of KOITER's shell theory

Let $\mathcal{O}$ be a bounded domain of the plane and let $\mathbf{\Phi}$ be a mapping : $\mathcal{O} \mapsto S \subset \mathbb{R}^3$ of class $C^3(\mathcal{O})$. Here S represents a surface in $\mathbb{R}$ determined by the functions $x^i = x^i(\xi)$, $\xi = (\xi^1, \xi^2) \in \mathcal{O}$. We have $\mathbf{\Phi} = x^i(\xi)\mathbf{e}_i$, where $\mathbf{e}_i$ are versors of the Cartesian system $\{x^i\}$; $\mathbf{e}_i.\mathbf{e}_j = \delta_{ij}$. The vectors tangent to the lines ξ^α on S are given by

$$\mathbf{a}_\alpha = \frac{\partial \mathbf{\Phi}}{\partial \xi^\alpha} = \mathbf{\Phi}_{,\alpha}. \tag{2.1}$$

Let

$$\mathbf{a}_3 = \frac{\mathbf{a}_1 \times \mathbf{a}_2}{|\mathbf{a}_1 \times \mathbf{a}_2|} \tag{2.2}$$

be a vector of unit length normal to S.
Let us define the components of the metric tensor

$$a_{\alpha\beta} = \mathbf{a}_\alpha.\mathbf{a}_\beta \tag{2.3}$$

and of the curvature tensor

$$b_{\alpha\beta} = -\mathbf{a}_\alpha.\mathbf{a}_{3,\beta} = \mathbf{a}_3.\mathbf{a}_{\alpha,\beta}. \tag{2.4}$$

The Christoffel symbols are defined by

$$\Gamma^\alpha_{\beta\gamma} = a^{\alpha\lambda}\Gamma_{\lambda\beta\gamma}, \quad \Gamma_{\alpha\beta\gamma} = \frac{1}{2}(a_{\alpha\beta,\gamma} + a_{\alpha\gamma,\beta} - a_{\beta\gamma,\alpha}) \ , \tag{2.5}$$

where $a^{\alpha\gamma}a_{\gamma\beta} = \delta^\alpha_\beta$, $\mathbf{a}^\alpha = a^{\alpha\gamma}\mathbf{a}_\gamma$. Then

$$\mathbf{a}_{\alpha,\beta} = \Gamma^\sigma_{\alpha\beta}\mathbf{a}_\sigma + b_{\alpha\beta}\mathbf{a}_3 \tag{2.6}$$

which is called the Gauss formula. Let us recall the formula of Weingarten

$$\mathbf{a}_{3,\alpha} = -b^\lambda_\alpha \mathbf{a}_\lambda. \tag{2.7}$$

The surfacial covariant derivative of a vector (T_α) will be defined by

$$T_{\alpha|\beta} = T_{\alpha,\beta} - \Gamma^{\lambda}_{\alpha\beta} T_\lambda. \tag{2.8}$$

Note that $T_\alpha = w|_\alpha$ defines a vector if w is a scalar. The Laplace operator assumes the form

$$\Delta w = w|_{\alpha\beta} a^{\alpha\beta} \tag{2.9a}$$

or

$$\Delta w = w|_{\alpha\cdot}^{\ \alpha} = w|_{\cdot\alpha}^{\alpha}\ . \tag{2.9b}$$

Specifically

$$\Delta w = (w_{,\alpha\beta} - \Gamma^{\sigma}_{\alpha\beta} w_{,\sigma}) a^{\alpha\beta}. \tag{2.10}$$

The surfacial covariant derivative has the following property

$$T_{\alpha|\beta\gamma} - T_{\alpha|\gamma\beta} = R_{\lambda\alpha\beta\gamma} T^{\lambda}, \tag{2.11}$$

with the so-called mixed curvature tensor given by

$$R_{\lambda\alpha\beta\gamma} = b_{\alpha\gamma} b_{\beta\lambda} - b_{\alpha\beta} b_{\lambda\gamma}. \tag{2.12}$$

We say that the system (ξ^α) is orthogonal, if

$$a_{12} = 0. \tag{2.13}$$

Then the vectors $\mathbf{a}_\alpha$ and $\mathbf{a}^\alpha$ are co-linear and the so-called physical components of vectors and tensors are defined uniquely.
For instance, if $\mathbf{T} = T_\alpha \mathbf{a}^\alpha + T_3 \mathbf{a}^3$, then

$$\mathbf{T} = T_1^* \mathbf{a}_1^* + T_2^* \mathbf{a}_2^* + T_3^* \mathbf{a}_3, \tag{2.14}$$

with

$$\mathbf{a}_\alpha^* = \mathbf{a}_\alpha / |\mathbf{a}_\alpha|, \quad \mathbf{a}_3 = \mathbf{a}^3 \tag{2.15}$$

and

$$T_\alpha^* = \frac{T_\alpha}{\sqrt{a_{\alpha\alpha}}}, \quad T_3^* = T_3 = T^3 \tag{3.16}$$

(do not sum over α).

If $\mathbf{T} = T_{\alpha\beta}\mathbf{a}^\alpha \otimes \mathbf{a}^\beta$, then

$$\mathbf{T} = T^*_{11}\mathbf{a}^*_1 \otimes \mathbf{a}^*_1 + T^*_{12}\mathbf{a}^*_1 \otimes \mathbf{a}^*_2 + T^*_{21}\mathbf{a}^*_2 \otimes \mathbf{a}^*_1 + T^*_{22}\mathbf{a}^*_2 \otimes \mathbf{a}^*_2 \tag{2.17}$$

and

$$T^*_{\alpha\beta} = \frac{T_{\alpha\beta}}{\sqrt{a_{\alpha\alpha}}\sqrt{a_{\beta\beta}}}. \tag{2.18}$$

The physical components are referred to unit vectors $\mathbf{a}^*_\alpha$ and that is why they have correct units.
The state of displacement of the shell is determined by the vector field

$$\mathbf{u} = u_\alpha\mathbf{a}^\alpha + w\mathbf{a}^3 \tag{2.19}$$

which fixes the deformed configuration of the middle surface S.
The in-plane deformations of the shell are defined by

$$\gamma_{\alpha\beta}(\mathbf{u}) = \frac{1}{2}(u_{\alpha|\beta} + u_{\beta|\alpha}) - b_{\alpha\beta}w, \tag{2.20}$$

while the changes of curvature of the shell middle surface are determined by

$$\rho_{\alpha\beta}(\mathbf{u}) = w|_{\alpha\beta} - b^\lambda_\alpha b_{\lambda\beta}w + b^\lambda_{\alpha|\beta}u_\lambda + b^\lambda_\alpha u_{\lambda|\beta} + b^\lambda_\beta u_{\lambda|\alpha}. \tag{2.21}$$

Note that $\gamma_{\alpha\beta} = \gamma_{\beta\alpha}$ and $\rho_{\alpha\beta} = \rho_{\beta\alpha}$, since $b^\lambda_{\alpha|\beta} = b^\lambda_{\beta|\alpha}$ by the Codazzi-Mainardi relations.
Let us recall the tensor of reduced plane-stress moduli for an isotropic body

$$C^{\alpha\beta\lambda\mu} = \frac{E}{1-\nu^2}\left[\nu a^{\alpha\beta}a^{\lambda\mu} + \frac{1-\nu}{2}(a^{\alpha\lambda}a^{\beta\mu} + a^{\alpha\mu}a^{\beta\lambda})\right]; \tag{2.22}$$

here E represents Young's modulus and ν is the Poisson ratio.
Let the shell thickness be constant and equal h. The membrane and bending stiffnesses are defined by

$$A^{\alpha\beta\lambda\mu} = hC^{\alpha\beta\lambda\mu}, \quad D^{\alpha\beta\lambda\mu} = \frac{h^3}{12}C^{\alpha\beta\lambda\mu}. \tag{2.23}$$

The membrane forces $\mathcal{N}^{\alpha\beta}$ and moments $M^{\alpha\beta}$ are interrelated with strains (2.20)-(2.21) by the first-order constitutive relations of Love

$$\mathcal{N}^{\alpha\beta} = A^{\alpha\beta\lambda\mu}\gamma_{\lambda\mu}(\mathbf{u}), \quad M^{\alpha\beta} = D^{\alpha\beta\lambda\mu}\rho_{\lambda\mu}(\mathbf{u}). \tag{2.24}$$

Note that $\mathcal{N}^{\alpha\beta} = \mathcal{N}^{\beta\alpha}$, $M^{\alpha\beta} = M^{\beta\alpha}$.
Assume that the shell is clamped along $\partial S = \Gamma$. Then

$$\mathbf{u} = \mathbf{0} \ \text{ and } \ \frac{\partial w}{\partial n} = 0 \ \text{ on } \ \Gamma, \tag{2.25}$$

where $\mathbf{n}$ represents a vector of unit length normal to Γ and tangent to S. Thus any

$$\mathbf{u} \in V_0 = \left(H_0^1(\mathcal{O})\right)^2 \times H_0^2(\mathcal{O}) \tag{2.26}$$

satisfies the conditions (2.25).
Assume that a clamped shell is subject to a surface loading $\mathbf{p} = (p^\alpha, p^3)$. The variational equilibrium equation has the form

$$\int_{\mathcal{O}} [\mathcal{N}^{\alpha\beta}\gamma_{\alpha\beta}(\mathbf{v}) + M^{\alpha\beta}\rho_{\alpha\beta}(\mathbf{v})]\sqrt{a}d\xi = \int_{\mathcal{O}} p^i v_i \sqrt{a}d\xi \quad \forall \mathbf{v} \in V_0, \tag{2.27}$$

where $\mathbf{v} = (v_1, v_2, v)$, $a = \det(a_{\alpha\beta})$, $d\xi = d\xi^1 d\xi^2$.
Let us define the bilinear form

$$a_1(\mathbf{u}, \mathbf{v}) = \int_{\mathcal{O}} [A^{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}(\mathbf{u})\gamma_{\lambda\mu}(\mathbf{v}) + D^{\alpha\beta\lambda\mu}\rho_{\alpha\beta}(\mathbf{u})\rho_{\lambda\mu}(\mathbf{v})]\sqrt{a}d\xi \tag{2.28}$$

and the linear form

$$f_1(\mathbf{v}) = \int_{\mathcal{O}} p^i v_i \sqrt{a}d\xi. \tag{2.29}$$

Then the problem of equilibrium of a clamped shell is formuled as follows

$$(\mathrm{P}_1)\begin{cases} \text{find } \ \mathbf{u} \in V_0 \ \text{ such that} \\ a_1(\mathbf{u}, \mathbf{v}) = f_1(\mathbf{v}) \quad \forall \mathbf{v} \in V_0. \end{cases} \tag{2.30}$$

This problem is uniquely solvable, see Bernadou (1995).
Assume that a segment $\Gamma_\sigma \subset \Gamma$ is loaded and Γ_u is clamped. Then the space of admissible displacements is given by

$$V_1 = \left\{\mathbf{v} \in \left(H^1(\mathcal{O})\right)^2 \times H^2(\mathcal{O}) \mid \mathbf{v} = \mathbf{0}, \frac{\partial v}{\partial n} = 0 \ \text{ on } \ \Gamma_u\right\}. \tag{2.31}$$

The equilibrium equation (2.27) assumes the form

$$\int_{\mathcal{O}} [\mathcal{N}^{\alpha\beta}\gamma_{\alpha\beta}(\mathbf{v}) + M^{\alpha\beta}\rho_{\alpha\beta}(\mathbf{v})]\sqrt{a}d\xi = f_1(\mathbf{v}) + f_2(\mathbf{v}) \quad \forall \mathbf{v} \in V_1 \tag{2.32}$$

with

$$f_2(\mathbf{v}) = \int_{\Gamma_\sigma} \Big(\widehat{T}^\alpha v_\alpha + \widehat{Q} v + \widehat{M}_B \frac{\partial v}{\partial n}\Big) ds, \tag{2.33}$$

where $\widehat{T}^\alpha$, $\widehat{Q}$, $\widehat{M}_B$ are given along Γ_σ. They are certain averages of the boundary loading, the averages taken along the thickness. For their definitions, see Niordson (1985). It is assumed here only that these quantities are known along Γ_σ.

The boundary value problem (P_1) should be replaced with (P_2) with f_1 replaced by $f_1 + f_2$ and V_0 replaced by V_1. This new problem is also uniquely solvable, provided that meas $(\Gamma_u) > 0$.

In the case of $\Gamma = \Gamma_\sigma$ the problem (P_2) is still well-posed, if

$$(f_1 + f_2)(\mathbf{v}) = 0 \quad \forall \mathbf{v} \in \mathcal{R} \tag{2.34}$$

where

$$\mathcal{R} = \{\mathbf{v} \in (H^1(\mathcal{O}))^2 \times H^2(\mathcal{O}) \mid a_1(\mathbf{v}, \mathbf{v}) = 0\}. \tag{2.35}$$

According to the theorem of Bernadou and Ciarlet (see Bernadou (1995)) the set $\mathcal{R}$ is composed of translations and infinitesimal rotations

$$\mathcal{R} = \{\mathbf{v} \mid \exists \Omega_0 \ \text{ and } \ \mathbf{v}_0 \ \text{ such that } \ \mathbf{v} = \mathbf{v}_0 + \Omega_0 \times \mathbf{\Phi}(\xi)\}. \tag{2.36}$$

Here $\mathbf{v}_0$, Ω_0 are vectors in $\mathbb{R}^3$. The condition (2.34) means that the loading applied is self-equilibrated : the resultant force vector and moment vector should vanish. The solution $\mathbf{u}$ is determined up to the fields from $\mathcal{R}$, i.e. up to rigid translations and rigid infinitesimal rotations.

The equilibrium equation (2.32) implies the local equilibrium equations (which will not be recalled) and the subsidiary conditions along Γ_σ

$$T^\alpha = \widehat{T}^\alpha, \ \ Q = \widehat{Q}, \ \ M_B = \widehat{M}_B \ \text{ on } \ \Gamma_\sigma, \tag{2.37}$$

where

$$\begin{aligned} &T^\alpha = (\mathcal{N}^{\beta\alpha} + 2b^\alpha_\gamma M^{\beta\gamma}) n_\beta, \\ &Q = -M^{\alpha\beta}|_\alpha n_\beta - \tfrac{\partial}{\partial s}(M^{\alpha\beta} n_\alpha t_\beta), \\ &M_B = M^{\alpha\beta} n_\alpha n_\beta, \end{aligned} \tag{2.38}$$

where $\mathbf{n} = n_\alpha \mathbf{a}^\alpha$, $\mathbf{t} = t_\alpha \mathbf{a}^\alpha$, $\mathbf{n}.\mathbf{t} = 0$, $|\mathbf{t}| = |\mathbf{n}| = 1$; $\partial/\partial s = (\partial/\partial \xi^\alpha) t^\alpha$; s parametrizes Γ_σ. We have

$$n^\alpha = \epsilon^{\alpha\beta} t_\beta, \ \ t^\sigma = \epsilon^{\alpha\sigma} n_\alpha, \tag{2.39}$$

where $(\epsilon^{\alpha\beta})$ are components of the Ricci tensor on the surface S.

3. KOITER-NIORDSON theory of a thin elastic spherical shell

An approximate character of the constitutive relations (2.24) enables one to modify them by
i) adding to $\mathcal{N}^{\alpha\beta}$ the terms of order $O(M^{\alpha\beta}/R)$,
ii) adding to $\rho_{\lambda\mu}$ the terms of order $O(\gamma_{\alpha\beta}/R)$, where $R = \min(R_\sigma)$, R_σ are radii of principal curvatures of S.
This statement has been justified by Koiter (1960) and used by Koiter (1963) and Niordson (1985, Chapter 13) to introduce a new simplified theory of thin spherical shells. This new theory is nevertheless not less accurate than the theory recalled in Sec. 2.
The aim of this section is to report this specific theory, applicable only to spherical shells.

3.1. General equations

The setting below is mainly based on the Chapter 13 of the book by Niordson (1985).
Let the centre of the shell be in the point $(0,0,0)$ of the global Cartesian system $\{x^i\}$. The position vector $\mathbf{\Phi}$ of points on the middle surface of a spherical shell is determined by

$$\mathbf{\Phi} = r^i \mathbf{e}_i, \quad i = 1,2,3 \tag{3.1}$$

and

$$r^1 = R\sin\theta\cos\phi, \quad r^2 = R\sin\theta\sin\phi, \quad r^3 = R\cos\theta\ ;$$

here R represents the radius of the shell and $\xi^1 = \theta$, $\xi^2 = \phi$ are spherical coordinates on the surface $|\mathbf{\Phi}| = R = \text{const}$, cf. Fig. 3.1
Let us find the local basis $\mathbf{a}_i$ by (2.1) :

$$\begin{aligned}
\mathbf{a}_1 &= R[\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta],\\
\mathbf{a}_2 &= R[-\sin\theta\sin\phi, \sin\theta\cos\phi, 0],\\
\mathbf{a}_3 &= [\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta].
\end{aligned} \tag{3.2}$$

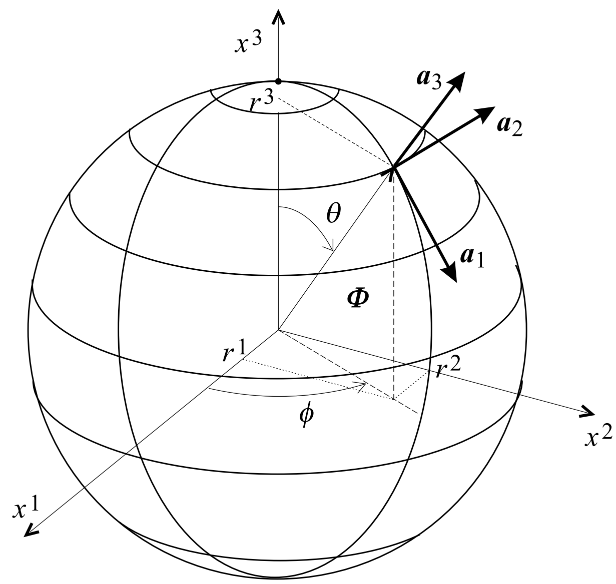

Figure 1: *Fig. 3.1. Parametrization of the middle surface of a spherical shell*

Hence the components of the metric tensor $(a_{\alpha\beta})$ are

$$
\begin{aligned}
&a_{11} = \mathbf{a}_1.\mathbf{a}_1 = R^2, \quad a_{12} = \mathbf{a}_1.\mathbf{a}_2 = 0, \\
&a_{22} = \mathbf{a}_2.\mathbf{a}_2 = R^2 \sin^2\theta
\end{aligned}
\tag{3.3}
$$

and the components of the curvature tensor

$$
b_{\alpha\beta} = \mathbf{a}_3.\mathbf{a}_{\alpha,\beta} \tag{3.4}
$$

are given by

$$
b_{11} = -R, \quad b_{12} = 0, \quad b_{22} = -R\sin^2\theta. \tag{3.5}
$$

The minus signs above are the consequence of directing the vector $\mathbf{a}_3$ along $\mathbf{\Phi}$, see Fig. 3.1. By comparing (3.3) with (3.5) we conclude that

$$
b_{\alpha\beta} = -\frac{1}{R} a_{\alpha\beta}. \tag{3.6}
$$

By using (2.5) one can check that

$$
\Gamma^2_{12} = \Gamma^2_{21} = \operatorname{ctg}\theta, \quad \Gamma^1_{22} = -\sin\theta\cos\theta \tag{3.7}
$$

and other components of $\mathbf{\Gamma}$ vanish. The Laplace-Beltrami operator is defined by (2.9). In the spherical coordinate system this operator assumes the form

$$\Delta w = \frac{1}{R^2}\Big(\frac{\partial^2 w}{\partial\theta^2} + \operatorname{ctg}\theta\frac{\partial w}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 w}{\partial\phi^2}\Big). \tag{3.8}$$

The surfacial covariant derivative (2.8) has the property (2.11). Here

$$R_{\lambda\alpha\beta\gamma} = \frac{1}{R^2}(a_{\alpha\gamma}a_{\beta\lambda} - a_{\alpha\beta}a_{\lambda\gamma}). \tag{3.9}$$

For $T_\alpha = w_{,\alpha} = w|_\alpha$ we have $T_{\alpha|\beta} = T_{\beta|\alpha}$, but $w|_{\gamma\beta\alpha} \neq w|_{\gamma\alpha\beta}$. We have, in particular

$$w|_{\alpha\gamma\cdot}{}^{\gamma} = (\Delta w)|_\alpha + \frac{1}{R^2}w_{,\alpha} \tag{3.10}$$

which follows from (2.11). This formula will be helpful in finding the operator defining the effective shear force Q on the boundary of the shell.
By (3.6) we have got now

$$\begin{aligned}\gamma_{\alpha\beta}(\mathbf{u}) &= \tfrac{1}{2}(u_{\alpha|\beta} + u_{\beta|\alpha}) + \tfrac{1}{R}a_{\alpha\beta}w,\\ \rho_{\alpha\beta}(\mathbf{u}) &= w|_{\alpha\beta} - \tfrac{1}{R}(u_{\beta|\alpha} + u_{\alpha|\beta}) - \tfrac{1}{R^2}a_{\alpha\beta}w,\end{aligned} \tag{3.11}$$

which makes it possible to rearrange the energy density

$$\mathcal{N}^{\alpha\beta}\gamma_{\alpha\beta}(\mathbf{v}) + M^{\alpha\beta}\rho_{\alpha\beta}(\mathbf{v}) \tag{3.12}$$

to the form

$$\widetilde{N}^{\alpha\beta}\gamma_{\alpha\beta}(\mathbf{v}) + M^{\alpha\beta}\tilde{\rho}_{\alpha\beta}(v) \tag{3.13}$$

with

$$\widetilde{N}^{\alpha\beta} = \mathcal{N}^{\alpha\beta} - \frac{2}{R}M^{\alpha\beta}, \quad \widetilde{N}^{\alpha\beta} = \widetilde{N}^{\beta\alpha}, \tag{3.14}$$

$$\tilde{\rho}_{\alpha\beta}(v) = v|_{\alpha\beta} + \frac{1}{R^2}a_{\alpha\beta}v, \quad \tilde{\rho}_{\alpha\beta} = \tilde{\rho}_{\beta\alpha}. \tag{3.15}$$

Note that $\mathbf{v} = (v_1, v_2, v)$ and $\tilde{\rho}$ depends only on v.
The effective forces along Γ_σ, see (2.38), assume the form

$$T^\alpha = \widetilde{N}^{\alpha\beta}n_\beta \tag{3.16}$$

and the formulae for Q and M_B remain unchanged.
According to the remarks (i), (ii) at the beginning of this section the constitutive relations (2.24) may be replaced with

$$\widetilde{N}^{\alpha\beta} = A^{\alpha\beta\lambda\mu}\gamma_{\lambda\mu}(\mathbf{u}), \quad M^{\alpha\beta} = D^{\alpha\beta\lambda\mu}\tilde{\rho}_{\lambda\mu}(w) \tag{3.17}$$

and this is the crucial point of the Koiter-Niordson modelling of thin elastic spherical shells.
Let us define the bilinear form

$$a_2(\mathbf{u},\mathbf{v}) = \int_{\mathcal{O}}[A^{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}(\mathbf{u})\gamma_{\lambda\mu}(\mathbf{v}) + D^{\alpha\beta\lambda\mu}\tilde{\rho}_{\alpha\beta}(w)\tilde{\rho}_{\lambda\mu}(v)]\sqrt{a}d\xi. \tag{3.18}$$

Note that the space

$$\mathcal{R} = \{\mathbf{v} \in (H^1(\mathcal{O}))^2 \times H^2(\mathcal{O} \mid a_2(\mathbf{v},\mathbf{v}) = 0\} \tag{3.19}$$

is equal to $\mathcal{R}$ defined by (2.35) and hence is given by (2.36). This property is due to the relationship

$$\tilde{\rho}_{\alpha\beta}(v) = \rho_{\alpha\beta}(\mathbf{v}) + \frac{2}{R}\gamma_{\alpha\beta}(\mathbf{v}), \tag{3.20}$$

by which we see that

$$\gamma_{\alpha\beta}(\mathbf{v}) = 0 \wedge \rho_{\alpha\beta}(\mathbf{v}) = 0$$

if and only if

$$\gamma_{\alpha\beta}(\mathbf{v}) = 0 \wedge \tilde{\rho}_{\alpha\beta}(v) = 0.$$

One can prove also that the form $a_2(.,.)$ is V_1-elliptic and continuous. Thus the problem

$$(\widetilde{\mathrm{P}}) \begin{cases} \text{find } \mathbf{u} \in V_1 \text{ such that} \\ a_2(\mathbf{u},\mathbf{v}) = (f_1 + f_2)(\mathbf{v}) \quad \forall \mathbf{v} \in V_1. \end{cases} \tag{3.21}$$

is uniquely solvable.
By using the relations

$$\begin{aligned} &v_\alpha = v_n n_\alpha + v_t t_\alpha, \quad v_n = v_\alpha n^\alpha, \quad v_t = v_\alpha t^\alpha, \\ &N = \widetilde{N}^{\alpha\beta} n_\alpha n_\beta, \quad S = \widetilde{N}^{\alpha\beta} n_\alpha t_\beta, \\ &\widehat{N} = \widehat{T}^\alpha n_\alpha, \quad \widehat{S} = \widehat{T}^\alpha t_\alpha \end{aligned} \tag{3.22}$$

one can rearrange $f_2(\mathbf{v})$ to the form

$$f_2(\mathbf{v}) = \int_{\Gamma_\sigma} \Big(\widehat{N}v_n + \hat{S}v_t + \hat{Q}v + \widehat{M}_B \frac{\partial v}{\partial n}\Big) ds \tag{3.23}$$

and the conditions (2.37) are replaced with

$$N = \widehat{N}, \quad S = \hat{S}, \quad Q = \hat{Q}, \quad M_B = \widehat{M}_B. \tag{3.24}$$

The variational equation of equilibrium :

$$\int_{\mathcal{O}} (\widetilde{N}^{\alpha\beta}\gamma_{\alpha\beta}(\mathbf{v}) + M^{\alpha\beta}\tilde{\rho}_{\alpha\beta}(\mathbf{v}))\sqrt{a}d\xi = f_1(\mathbf{v}) + f_2(\mathbf{v}) \quad \forall \mathbf{v} \in V_1 \tag{3.25}$$

implies the local equilibrium equations of the form

$$\begin{aligned} &\widetilde{N}^{\alpha\beta}|_\beta + p^\alpha = 0, \\ &M^{\alpha\beta}|_{\alpha\beta} + \tfrac{1}{R^2}M^{\alpha}_{\cdot\alpha} + \tfrac{1}{R}\widetilde{N}^{\alpha}_{\cdot\alpha} = p^3 \end{aligned} \tag{3.26}$$

and the subsidiary boundary conditions (3.24) on Γ_σ.
One can check that the strain measures $\gamma_{\alpha\beta}(\mathbf{u})$ and $\tilde{\rho}_{\alpha\beta}(w)$ satisfy the following three compatibility relations

$$\begin{aligned} &\Big(\Delta + \tfrac{1}{R^2}\Big)\gamma_{\sigma\cdot}^{\cdot\sigma} - \gamma^{\alpha\cdot\ \ \beta}_{\cdot\beta|\alpha\cdot} - \tfrac{1}{R}\tilde{\rho}_{\sigma\cdot}^{\cdot\sigma} = 0, \\ &\tilde{\rho}_{\beta\gamma|\alpha} - \tilde{\rho}_{\alpha\gamma|\beta} = 0. \end{aligned} \tag{3.27}$$

3.2. Reducing of the problem to two equations involving two scalar unknowns

Assume that $\widetilde{N}^{\alpha\beta}$ are represented by

$$\widetilde{N}^{\alpha\beta} = \epsilon^{\alpha\sigma}\epsilon^{\beta\eta}\widetilde{\Phi}|_{\eta\sigma} + \frac{a^{\alpha\beta}}{R^2}\widetilde{\Phi}. \tag{3.28}$$

One can prove that if $\widetilde{\Phi}(\xi)$ is sufficiently regular, then the equations $\widetilde{N}^{\alpha\beta}|_\beta = 0$ are satisfied identically. Let $F^{\alpha\beta}$ be particular integral for $(3.26)_1$. Let us put $\widetilde{\Phi} = EhR\Phi$ to make the units of Φ and w the same. It is seen that the local equations (3.26) reduce to one equation $(3.26)_3$ since the two first equations are identically satisfied if we put

$$\widetilde{N}^{\alpha\beta} = EhR\Big[\epsilon^{\alpha\sigma}\epsilon^{\beta\eta}\Phi|_{\eta\sigma} + \frac{1}{R^2}a^{\alpha\beta}\Phi\Big] + F^{\alpha\beta}. \tag{3.29}$$

Let us substitute (2.22) into (3.17), remembering that $(2.23)_2$ holds. We find

$$M^{\alpha\beta} = D[(1-\nu)\tilde{\rho}^{\alpha\beta}(w) + \nu a^{\alpha\beta}\tilde{\rho}_{\lambda\cdot}^{\cdot\lambda}(w)], \tag{3.30}$$

where $D = Eh^3/(12(1-\nu^2))$ represents the bending stiffness of the shell. Thus by (3.15) and (2.9) we have

$$M^{\alpha\beta} = D\Big[(1-\nu)\Big(w|^{\alpha\beta} + \frac{1}{R^2}a^{\alpha\beta}w\Big) + \nu a^{\alpha\beta}\Big(\Delta + \frac{2}{R^2}\Big)w\Big]. \tag{3.31}$$

Substitution of (3.31) and (3.29) into (3.26) gives

$$\Big(\Delta + \frac{2}{R^2}\Big)\Big[\Big(\Delta + \frac{1+\nu}{R^2}\Big)w + \frac{1-\nu^2}{kR^2}\Phi\Big] = \frac{1}{D}\Big[p^3 - \frac{1}{R}F^{\alpha}_{\cdot\alpha}\Big]\,, \tag{3.32}$$

where $k = h^2/12R^2$.

Let us proceed now to express the compatibility equation $(3.27)_1$ in terms of $\widetilde{N}^{\alpha\beta}$. We invert the constitutive relation $(3.17)_1$:

$$\gamma^{\alpha\beta} = \frac{1}{Eh}[(1+\nu)\widetilde{N}^{\alpha\beta} - \nu a^{\alpha\beta}\widetilde{N}^{\alpha}_{\cdot\alpha}] \tag{3.33}$$

and insert it into $(3.27)_1$. On using the representation (3.29) one finds

$$\Big(\Delta + \frac{2}{R^2}\Big)\Big[\Big(\Delta + \frac{1-\nu}{R^2}\Big)\Phi - \frac{w}{R^2}\Big] = 0. \tag{3.34}$$

Thus we have arrived at two scalar equations (3.32) and (3.34) for two scalar unknowns.

Niordson proved that the solution to the system (3.32), (3.34) in the case of $p^i = 0$ is given by

$$\begin{aligned} w &= -\tfrac{1-\nu^2}{kR^2}\Xi + w^{(1)}, \\ \Phi &= \Big(\Delta + \tfrac{1+\nu}{R^2}\Big)\Xi, \end{aligned} \tag{3.35}$$

where Ξ satisfies

$$\Big(\Delta+\frac{2}{R^2}\Big)\Big[\Big(\Delta+\frac{1+\nu}{R^2}\Big)\Big(\Delta+\frac{1-\nu}{R^2}\Big)+\frac{1-\nu^2}{kR^4}\Big]\Xi=0 \tag{3.36}$$

and $w^{(1)}$ fulfills the equations :

$$\Big(\Delta+\frac{2}{R^2}\Big)w^{(1)}=0,\quad \gamma_{\alpha\beta}(u_1^{(1)},u_2^{(1)},w^{(1)})=0. \tag{3.37}$$

It turns out later that the conditions $\gamma_{\alpha\beta}(\mathbf{u}^{(1)})=0$ imply $\Big(\Delta+\frac{2}{R^2}\Big)w^{(1)}=0$, hence the first equation of (3.37) may be omitted as redundant.
Let us rewrite (3.36) in the form

$$\Big(\Delta+\frac{2}{R^2}\Big)\Big\{\Delta^2+\frac{2}{R^2}\Delta+\frac{1}{R^4}\Big[1+\frac{12R^2}{h^2}(1-\nu^2)-\nu^2\Big]\Big\}\Xi=0 \tag{3.38}$$

and factorize it to the form

$$\Big(\Delta+\frac{2}{R^2}\Big)\Big(\Delta+\frac{1+2i\tilde{\rho}^2}{R^2}\Big)\Big(\Delta+\frac{1-2i\tilde{\rho}^2}{R^2}\Big)\Xi=0 \tag{3.39}$$

with

$$2\tilde{\rho}^2=\Big[\frac{12R^2}{h^2}(1-\nu^2)-\nu^2\Big]^{1/2} \tag{3.40a}$$

since $(R/h)^2\gg 1$ and $\nu\in(-1,1/2)$, but practically $\nu\in(0,1/2)$, one can replace $\tilde{\rho}^2$ with

$$\rho^2=\sqrt{3(1-\nu^2)}\frac{R}{h}. \tag{3.40b}$$

A general solution of (3.39) can be represented as a sum

$$\Xi=\Xi_1+\Xi_2+\Xi_3, \tag{3.41}$$

where

$$\Big(\Delta+\frac{2}{R^2}\Big)\Xi_1=0, \tag{3.42}$$

$$\Big(\Delta+\frac{1+2i\rho^2}{R^2}\Big)\Xi_4=0, \tag{3.43}$$

and

$$\Xi_2 = Re(\Xi_4), \ \ \Xi_3 = Im(\Xi_4).$$

The boundary conditions (3.24) may be expressed in terms of w and Φ. It is sufficient to substitute (3.29) and (3.31) into the expressions for N, S, Q and M_B given by (3.22) and (2.38) and make use of (2.39). Thus we find

$$\begin{aligned} N &= F^{\alpha\beta} n_\alpha n_\beta + EhR\Big[t^\alpha t^\beta \Phi|_{\beta\alpha} + \tfrac{1}{R^2}\Phi\Big], \\ S &= -EhRt^\alpha n^\beta \Phi|_{\beta\alpha}, \\ M_B &= D\Big[(1-\nu) n^\alpha n^\beta w|_{\alpha\beta} + \nu \Delta w + \tfrac{1+\nu}{R^2} w\Big], \\ Q &= -D\Big[n^\alpha\Big(\Big(\Delta + \tfrac{2}{R^2}\Big)w\Big)\Big|_\alpha + (1-\nu)t^\alpha (n^\beta t^\gamma w|_{\beta\gamma})|_\alpha\Big]. \end{aligned} \tag{3.44}$$

To express u_1 and u_2 in terms of w and Φ one should integrate the equations (3.11). Then new integration constants appear.

3.3. Construction of the solution to the system (3.32)-(3.34)

Let us introduce the physical components of u, w according to the rules (2.16). We define

$$u = \frac{u_1}{R}, \ \ v = \frac{u_2}{R\sin\theta}, \tag{3.45a}$$

where $u = u_1^*$ and $v = u_2^*$, cf. Fig. 3.2. The physical components of the trial field $\mathbf{v} = (v_1, v_2, v)$ are defined by

$$\overline{u} = \frac{v_1}{R}, \ \ \overline{v} = \frac{v_2}{R\sin\theta}, \ \ \overline{w} = v. \tag{3.45b}$$

According to the rules (2.18) we define the physical components of $(\gamma_{\alpha\beta})$

$$\gamma^*_{\alpha\beta} = \varepsilon_{\alpha\beta} \tag{3.46}$$

and find

$$\begin{aligned} \varepsilon_{11} &= \tfrac{1}{R}\Big(\tfrac{\partial u}{\partial\theta} + w\Big), \\ \varepsilon_{22} &= \tfrac{1}{R\sin\theta}\Big(\tfrac{\partial v}{\partial\phi} + \cos\theta\ u + \sin\theta\ w\Big), \\ 2\varepsilon_{12} &= \tfrac{1}{R\sin\theta}\Big(\tfrac{\partial u}{\partial\phi} + \sin\theta\tfrac{\partial v}{\partial\theta} - \cos\theta\ v\Big). \end{aligned} \tag{3.47}$$

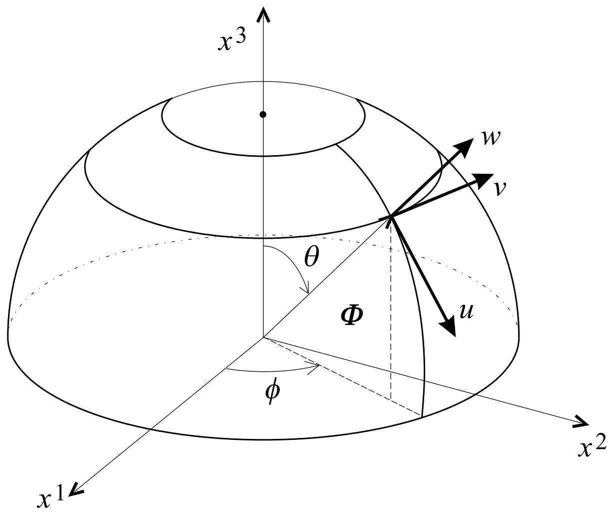

Figure 2: *Fig. 3.2. Directions of positive displacements : u, v, w*

According to (3.37) we must solve the system : $\varepsilon_{\alpha\beta} = 0$. We predict the solution in the form

$$\begin{aligned} u &= u_m(\theta)\cos m\phi, \\ v &= v_m(\theta)\sin m\phi, \\ w &= w_m(\theta)\cos m\phi. \end{aligned} \tag{3.48}$$

Hence we find the Legendre equation

$$\Big(\frac{d^2}{d\theta^2} + \operatorname{ctg}\theta\frac{d}{d\theta} + 2 - \frac{m^2}{\sin^2\theta}\Big)w_m(\theta) = 0 \tag{3.49}$$

or

$$\Big(\Delta + \frac{2}{R^2}\Big)(w_m(\theta)\cos m\phi) = 0 \tag{3.50}$$

and its solution has the form

a) $m = 0$

$$w_0(\theta) = A_0^{(0)}w_0^{(1)}(\theta) + B_0^{(0)}w_0^{(2)}(\theta),$$

where

$$w_0^{(1)}(\theta) = \cos\theta, \;\; w_0^{(2)}(\theta) = \cos\theta\ln\left|\operatorname{ctg}\frac{\theta}{2}\right| - 1. \tag{3.51}$$

b) $m = 1$

$$w_1(\theta) = A_1^{(1)} w_1^{(1)}(\theta) + B_1^{(1)} w_1^{(2)}(\theta),$$

where

$$w_1^{(1)}(\theta) = \sin\theta, \quad w_1^{(2)}(\theta) = \sin\theta \ln \left|\operatorname{tg}\frac{\theta}{2}\right| - \operatorname{ctg}\theta. \tag{3.52}$$

c) $m \geq 2$

$$w_m(\theta) = A_m(m + \cos\theta)\Big(\operatorname{tg}\frac{\theta}{2}\Big)^m + B_m(m - \cos\theta)(\operatorname{ctg}\frac{\theta}{2}\Big)^m \tag{3.53}$$

and $A_i^{(0)}$, $A_i^{(1)}$, A_m, $B_i^{(0)}$, $B_i^{(1)}$, B_m are constants.
One can show that the solutions $w_0^{(2)}$ and $w_1^{(2)}$ cannot satisfy the conditions $\varepsilon_{\alpha\beta} = 0$, hence are not admissible. The fields u_m, v_m associated with other solutions are

$$\begin{aligned} u_m &= \Big[-A_m\Big(\operatorname{tg}\tfrac{\theta}{2}\Big)^m + B_m\Big(\operatorname{ctg}\tfrac{\theta}{2}\Big)^m\Big]\sin\theta, \\ v_m &= \Big[-A_m\Big(\operatorname{tg}\tfrac{\theta}{2}\Big)^m - B_m\Big(\operatorname{ctg}\tfrac{\theta}{2}\Big)^m\Big]\sin\theta, \end{aligned} \tag{3.54}$$

$m = 0, 1, 2, \ldots$ We note that both the conditions (3.37) are fulfilled if (u, v, w) are of the form (3.53), (3.54) for $m \geq 0$. In particular, the condition (3.50) turned out to be a consequence of $\gamma_{\alpha\beta} = 0$, $\alpha, \beta \in \{1, 2\}$.
Concluding, the solution of (3.37) has the form

$$w^{(1)} = \sum_{m=0}^{\infty} [w_m(\theta)\cos m\phi + \tilde{w}_m(\theta)\sin m\phi] \tag{3.55}$$

with $w_m(\theta)$ given by (3.53) for $m \geq 0$ and $\tilde{w}_m(\theta)$ given by a similar formula with new constants $\widetilde{A}_m$, $\widetilde{B}_m$.
Let us turn now to solving the system (3.42) and (3.43). Solution to the problem (3.42) has the form :

$$\Xi_1 = \sum_{m=0}^{\infty} [C_m^{(1)}(\theta)\cos m\phi + D_m^{(1)}(\theta)\sin m\phi] \tag{3.56}$$

with $C_m^{(1)}(\theta)$, $D_m^{(1)}(\theta)$ satisfying (3.49) and (3.50).

Now, there is no argument to neglect the solution of type $w_0^{(2)}, w_1^{(2)}$, see (3.51).

The function $C_m^{(1)}(\theta)$ assumes the form :

a) $m = 0$

$$\begin{aligned} C_0^{(1)}(\theta) &= A_{3(0)} \cos\theta + A_{7(0)} T(\theta), \\ T(\theta) &= \cos\theta \ln \left|\operatorname{ctg} \tfrac{\theta}{2}\right| - 1. \end{aligned} \tag{3.57}$$

b) $m = 1$

$$\begin{aligned} C_1^{(1)}(\theta) &= A_{3(1)} \sin\theta + A_{7(1)} K(\theta), \\ K(\theta) &= \sin\theta \ln \left|\operatorname{tg} \tfrac{\theta}{2}\right| - \operatorname{ctg}\theta. \end{aligned} \tag{3.58}$$

c) $m \geq 2$

$$C_m^{(1)}(\theta) = A_{3(m)}(m + \cos\theta)\Big(\operatorname{tg}\frac{\theta}{2}\Big)^m + A_{7(m)}(m - \cos\theta)\Big(\operatorname{ctg}\frac{\theta}{2}\Big)^m \tag{3.59}$$

and the formulae for $D_m^{(1)}$ are similar.

Finding solutions to (3.43) is a much more difficult task. Assume that $\Xi_4(\theta, \phi) = y_m(\theta)\cos m\phi$ and change the variable θ into $x = \cos\theta$. Then equation (3.43) assumes the form

$$(1 - x^2)\frac{d^2 y_m}{dx^2} - 2x\frac{dy_m}{dx} + \left[\sigma(\sigma + 1) - \frac{m^2}{1 - x^2}\right] y_m = 0 \tag{3.60}$$

with

$$\sigma(\sigma + 1) = 1 + 2i\rho^2. \tag{3.61}$$

The quantity σ is a complex number

$$\sigma = a + ib, \tag{3.62}$$

both a and b being nonzero. The solution of (3.60) is spanned over the functions of Legendre type

$$P_\sigma^{-m}(\cos\theta) \quad \text{and} \quad P_\sigma^{-m}(-\cos\theta)\ , \tag{3.63}$$

because $\sigma \notin \mathbb{N}$, see Niordson (1985) or Wang and Guo (1989). The negative sign at m in (3.63) is taken for future convenience. In fact, $P_\sigma^m(x)$ and $P_\sigma^{-m}(x)$ differ in a factor.

Thus the solution of (3.43) is taken as linear combinations of real and imaginary parts of $P_\sigma^{-m}(\pm\cos\theta)$. We represent $\Xi_2+\Xi_3$ by

$$\Xi_2+\Xi_3=\sum_{m\geq 0}(\Xi_{(m)}^{(a)}\cos m\phi+\Xi_{(m)}^{(b)}\sin m\phi) \tag{3.64}$$

with

$$\Xi_{(m)}^{(a)}=A_{1(m)}R_\sigma^m(\theta)+A_{2(m)}S_\sigma^m(\theta)+A_{5(m)}R_\sigma^m(\pi-\theta)+A_{6(m)}S_\sigma^m(\pi-\theta) \tag{3.65}$$

and $\Xi_{(m)}^{(b)}$ is given by a similar formula. Here $R_\sigma^m(\theta)$ and $S_\sigma^m(\theta)$ are real and imaginary parts of $P_\sigma^{-m}(\cos\theta)$

$$P_\sigma^{-m}(\cos\theta)=R_\sigma^m(\theta)+iS_\sigma^m(\theta). \tag{3.66}$$

Note that $P_\sigma^{-m}(-\cos\theta)=R_\sigma^m(\pi-\theta)+iS_\sigma^m(\pi-\theta)$ and that is why the functions $R_\sigma^m(\pi-\theta)$ and $S_\sigma^m(\pi-\theta)$ enter (3.65).

The functions $R_\sigma^m(\theta)$ and $S_\sigma^m(\theta)$ are singular at $\theta=\pi$, while the functions $R_\sigma^m(\pi-\theta)$ and $S_\sigma^m(\pi-\theta)$ are singular at $\theta=0$. The functions involved in (3.65) cannot be easy expressed by known special functions. Their properties are reported in the Appendix.

By using (3.41), (3.56)-(3.59) and (3.64)-(3.65) one arrives at the final form of Ξ

$$\Xi(\theta,\phi)=\sum_{m=0}^{\infty}(\Xi_m^{(c)}(\theta)\cos m\phi+\Xi_m^{(s)}(\theta)\sin m\phi). \tag{3.67}$$

The indices "c" and "s" will below be omitted.

The functions $\Xi_m(\theta)$ assume three forms

a) $m=0$

$$\begin{aligned}\Xi_0(\theta)&=A_{1(0)}R_\sigma(\theta)+A_{2(0)}S_\sigma(\theta)+A_{5(0)}R_\sigma(\pi-\theta)+\\&+A_{6(0)}S_\sigma(\pi-\theta)+A_{3(0)}\cos\theta+A_{7(0)}T(\theta).\end{aligned} \tag{3.68}$$

Here $R_\sigma(\theta)=R_\sigma^0(\theta)$, $S_\sigma(\theta)=S_\sigma^0(\theta)$.

b) $m=1$

$$\begin{aligned}\Xi_1(\theta)&=A_{1(1)}R_\sigma^1(\theta)+A_{2(1)}S_\sigma^1(\theta)+A_{5(1)}R_\sigma^1(\pi-\theta)+\\&+A_{6(1)}S_\sigma^1(\pi-\theta)+A_{3(1)}\sin\theta+A_{7(1)}K(\theta).\end{aligned} \tag{3.69}$$

c) $m \geq 2$

$$\begin{aligned}\Xi_m(\theta) &= A_{1(m)} R_\sigma^m(\theta) + A_{2(m)} S_\sigma^m(\theta) + A_{5(m)} R_\sigma^m(\pi-\theta) + \\ &+ A_{6(m)} S_\sigma^m(\pi-\theta) + A_{3(m)}(m+\cos\theta)\Big(\operatorname{tg}\tfrac{\theta}{2}\Big)^m \\ &+ A_{7(m)}(m-\cos\theta)\Big(\operatorname{ctg}\tfrac{\theta}{2}\Big)^m.\end{aligned} \tag{3.70}$$

Remark 3.1. The case $m = 0$ means that the deformation is rotationally symmetric. The case $m = 1$ refers to the asymmetric loadings, causing essential reactions at supports. The cases $m \geq 2$ refer to higher harmonics describing self-equilibrated states of the loading. Thus all three cases refer to different states of the shell and it is reflected in the different forms of the solution Ξ.

Let us proceed now to finding the main scalar unknowns : w and Φ. They are determined by (3.35), (3.55) and (3.67). If we confine analysis to the terms with $\cos m\phi$ we find

$$\begin{aligned} w(\theta,\phi) &= \textstyle\sum_{m\geq 0} w_m(\theta)\cos m\phi, \\ \Phi(\theta,\phi) &= \textstyle\sum_{m\geq 0} \Phi_m(\theta)\cos m\phi, \end{aligned} \tag{3.71}$$

where

a) $m = 0$

$$\begin{aligned} w_0(\theta) = -\tfrac{1-\nu^2}{kR^2}\{&A_{1(0)} R_\sigma(\theta) + A_{2(0)} S_\sigma(\theta) + A_{5(0)} R_\sigma(\pi-\theta) + \\ &+ A_{6(0)} S_\sigma(\pi-\theta) + A_{4(0)}\cos\theta\}, \end{aligned} \tag{3.72a}$$

$$\begin{aligned} \Phi_0(\theta) &= \tfrac{1}{R^2}\{(\nu-1)A_{3(0)}\cos\theta + (\nu-1)A_{7(0)}T(\theta) + \\ &+ (\nu A_{1(0)} - 2\rho^2 A_{2(0)})R_\sigma(\theta) + (\nu A_{2(0)} + 2\rho^2 A_{1(0)})S_\sigma(\theta) \\ &+ (\nu A_{5(0)} - 2\rho^2 A_{6(0)})R_\sigma(\pi-\theta) + (\nu A_{6(0)} + 2\rho^2 A_{5(0)})S_\sigma(\pi-\theta)\}. \end{aligned} \tag{3.72b}$$

b) $m = 1$

$$\begin{aligned} w_1(\theta) = -\tfrac{1-\nu^2}{kR^2}\{&A_{1(1)} R_\sigma^1(\theta) + A_{2(1)} S_\sigma^1(\theta) + A_{5(1)} R_\sigma^1(\pi-\theta) + \\ &+ A_{6(1)} S_\sigma^1(\pi-\theta) + A_{4(1)}\sin\theta\}, \end{aligned} \tag{3.73a}$$

$$\begin{aligned} \Phi_1(\theta) &= \tfrac{1}{R^2}\{(\nu-1)A_{3(1)}\sin\theta + (\nu-1)A_{7(1)}K(\theta) + \\ &+ (\nu A_{1(1)} - 2\rho^2 A_{2(1)})R_\sigma^1(\theta) + (\nu A_{2(1)} + 2\rho^2 A_{1(1)})S_\sigma^1(\theta) \\ &+ (\nu A_{5(1)} - 2\rho^2 A_{6(1)})R_\sigma^1(\pi-\theta) + (\nu A_{6(1)} + 2\rho^2 A_{5(1)})S_\sigma^1(\pi-\theta)\}. \end{aligned} \tag{3.73b}$$

c) $m \geq 2$

$$w_m(\theta) = -\tfrac{1-\nu^2}{kR^2}\Big\{A_{1(m)}R_\sigma^m(\theta) + A_{2(m)}S_\sigma^m(\theta) + A_{5(m)}R_\sigma^m(\pi-\theta) + \\ + A_{6(m)}S_\sigma^m(\pi-\theta) + A_{4(m)}(m+\cos\theta)\Big(\operatorname{tg}\tfrac{\theta}{2}\Big)^m + A_{8(m)}(m-\cos\theta)\Big(\operatorname{ctg}\tfrac{\theta}{2}\Big)^m\Big\}, \tag{3.74a}$$

$$\Phi_m(\theta) = \tfrac{1}{R^2}\Big\{(\nu-1)A_{3(m)}(m+\cos\theta)\Big(\operatorname{tg}\tfrac{\theta}{2}\Big)^m + \\ + (\nu-1)A_{7(m)}(m-\cos\theta)\Big(\operatorname{ctg}\tfrac{\theta}{2}\Big)^m \\ + (\nu A_{1(m)} - 2\rho^2 A_{2(m)})R_\sigma^m(\theta) + (\nu A_{2(m)} + 2\rho^2 A_{1(m)})S_\sigma^m(\theta) \\ + (\nu A_{5(m)} - 2\rho^2 A_{6(m)})R_\sigma^m(\pi-\theta) + (\nu A_{6(m)} + 2\rho^2 A_{5(m)})S_\sigma^m(\pi-\theta)\Big\}. \tag{3.74b}$$

Remark 3.2. The expression $(13.74)_2$ for Φ in Niordson (1985) involves incorrect signs at ρ^2. Moreover, both the expressions (17.74) in Niordson (1985)are valid only for $m \geq 2$, which was not mentioned. Correct representations can be found in Czmoch and Nagórski (1979).

3.4. Solution in the case of a rotationally symmetric loading along one edge

Consider a segment of the spherical shell for

$$\theta \in [\theta_0, \pi/2], \quad \phi \in [0, 2\pi]. \tag{3.75}$$

Assume that the boundary $\theta = \pi/2$ is loaded by $\widehat{Q} = P$. Thus the boundary conditions to be fulfilled are

$$\begin{array}{llll} M_B(\theta_0) = 0, & (1) & M_B(\pi/2) = 0, & (4) \\ N(\theta_0) = 0, & (2) & N(\pi/2) = 0, & (5) \\ Q(\theta_0) = 0, & (3) & Q(\pi/2) = P, & (6) \end{array} \tag{3.76}$$

see Fig. 3.3

To find the set of equations for the integration constants one should substitute expressions for $w_0(\theta)$ and $\Phi_0(\theta)$ into the formulae (3.44).

Along $\theta = \pi/2$ we have $\mathbf{n} = \mathbf{a}_1$ and $\mathbf{t} = \mathbf{a}_2$, see Fig. 3.1. Using (3.7) one finds

$$n^\alpha n^\beta w|_{\beta\alpha} = \frac{1}{R^2}\frac{\partial^2 w}{\partial\theta^2}. \tag{3.77}$$

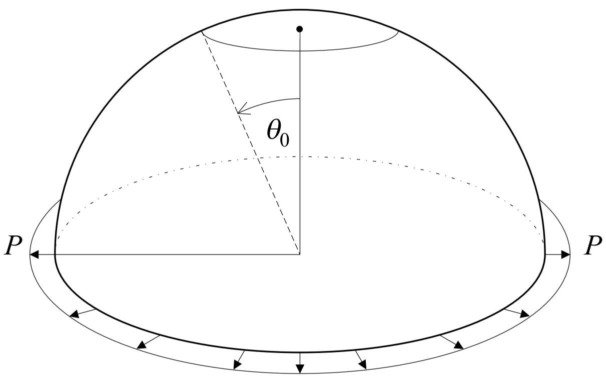

Figure 3: *Fig. 3.3. A hemisphere with an opening, loaded along the lower edge*

Appropriate differentiation of $w_0(\theta)$ can be performed with the help of the formulae summarized in the Appendix. We find eventually

$$M_B(\theta) = -\frac{Eh}{R^2} \sum_{i=1,2,4,5,6} A_i a_{Mi}(\theta), \tag{3.78}$$

where $A_i = A_{i(0)}$ and

$$\begin{aligned}
a_{M1}(\theta) &= -(1-\nu)\operatorname{ctg}\theta R_\sigma^{-1}(\theta) + 2\rho^2 S_\sigma(\theta) + \nu R_\sigma(\theta),\\
a_{M2}(\theta) &= -(1-\nu)\operatorname{ctg}\theta S_\sigma^{-1}(\theta) - 2\rho^2 R_\sigma(\theta) + \nu S_\sigma(\theta),\\
a_{M4}(\theta) &= 0,\\
a_{M5}(\theta) &= (1-\nu)\operatorname{ctg}\theta R_\sigma^{-1}(\pi-\theta) + 2\rho^2 S_\sigma(\pi-\theta) + \nu R_\sigma(\pi-\theta),\\
a_{M6}(\theta) &= (1-\nu)\operatorname{ctg}\theta S_\sigma^{-1}(\pi-\theta) - 2\rho^2 R_\sigma(\pi-\theta) + \nu S_\sigma(\pi-\theta).
\end{aligned} \tag{3.79}$$

Let us proceed to finding the expression for $N(\theta)$. By (3.7) we have

$$t^\alpha t^\beta \Phi|_{\beta\alpha} = (t^2)^2 \Phi|_{22} = \frac{1}{R^2}\Big[\frac{1}{\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} + \operatorname{ctg}\theta\frac{\partial\Phi}{\partial\theta}\Big]. \tag{3.80}$$

Thus we have

$$N = \frac{Eh}{R}\Big[\frac{1}{\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} + \operatorname{ctg}\theta\frac{\partial\Phi}{\partial\theta} + \Phi\Big]. \tag{3.81}$$

Since $\Phi = \Phi_0(\theta)$ we find

$$N = \frac{Eh}{R}\Big(\operatorname{ctg}\theta\frac{d\Phi_0}{d\theta} + \Phi_0\Big). \tag{3.82}$$

Substitution of (3.72b) and using some formulae for functions R_σ^m, S_σ^m gives

$$N = \frac{Eh}{R^3} \sum_{i=1,2,3,5,6,7} A_i a_{Ni}(\theta), \tag{3.83}$$

where

$$\begin{aligned}
a_{N1}(\theta) &= \nu(R_\sigma(\theta) + \operatorname{ctg}\theta R_\sigma^{-1}(\theta)) + 2\rho^2(S_\sigma(\theta) + \operatorname{ctg}\theta S_\sigma^{-1}(\theta)),\\
a_{N2}(\theta) &= \nu(S_\sigma(\theta) + \operatorname{ctg}\theta S_\sigma^{-1}(\theta)) - 2\rho^2(R_\sigma(\theta) + \operatorname{ctg}\theta R_\sigma^{-1}(\theta)),\\
a_{N3}(\theta) &= 0,\\
a_{N5}(\theta) &= \nu[R_\sigma(\pi-\theta) - \operatorname{ctg}\theta R_\sigma^{-1}(\pi-\theta)] + 2\rho^2[S_\sigma(\pi-\theta) - \operatorname{ctg}\theta S_\sigma^{-1}(\pi-\theta)],\\
a_{N6}(\theta) &= \nu[S_\sigma(\pi-\theta) - \operatorname{ctg}\theta S_\sigma^{-1}(\pi-\theta)] - 2\rho^2[R_\sigma(\pi-\theta) - \operatorname{ctg}\theta R_\sigma^{-1}(\pi-\theta)],\\
a_{N7}(\theta) &= \tfrac{1-\nu}{\sin^2\theta}.
\end{aligned} \tag{3.84}$$

To find the expression for Q we use $(3.44)_4$ and note that

$$n^\alpha f|_\alpha = \frac{1}{R}\frac{\partial f}{\partial\theta}, \tag{3.85}$$

since $n^1 = 1/R$ and $n^2 = 0$. Moreover

$$n^\beta t^\gamma w|_{\beta\gamma} = n^1 t^2 (w_{,21} - \Gamma_{21}^2 w|_2) = 0,$$

since $w = w_0(\theta)$. We find

$$Q = -\frac{D}{R}\frac{d}{d\theta}\Big(\Delta w_0 + \frac{2}{R^2} w_0\Big) \tag{3.86}$$

and hence

$$Q = \frac{Eh}{R^3} \sum_{i=1,2,5,6} A_i a_{Qi}(\theta) \tag{3.87}$$

with $A_i = A_{i(0)}$ and

$$\begin{aligned}
a_{Q1}(\theta) &= R_\sigma^{-1}(\theta) + 2\rho^2 S_\sigma^{-1}(\theta),\\
a_{Q2}(\theta) &= S_\sigma^{-1}(\theta) - 2\rho^2 R_\sigma^{-1}(\theta),\\
a_{Q5}(\theta) &= -R_\sigma^{-1}(\pi-\theta) - 2\rho^2 S_\sigma^{-1}(\pi-\theta),\\
a_{Q6}(\theta) &= -S_\sigma^{-1}(\pi-\theta) + 2\rho^2 R_\sigma^{-1}(\pi-\theta),\\
a_{Q4}(\theta) &= 0.
\end{aligned} \tag{3.88}$$

Now we are ready to express (3.76) in terms of the unknown constants $A_i = A_{i(0)}$. We normalize them by introducing $X_i = EhA_i/PR^3$. Note that $a_{Mi}(\pi/2) = a_{Ni}(\pi/2)$ for $i = 1, 2, 5, 6$. Thus the equations $N(\pi/2) = 0$ and $M_B(\pi/2) = 0$ differ in a term $(1-\nu)X_7$, which implies that $X_7 = 0$. Thus we have five equations : $M_B(\theta_0) = 0$, $N(\theta_0) = 0$, $Q(\theta_0) = 0$, $M_B(\pi/2) = 0$, $Q(\pi/2) = P$ and four unknowns : X_1, X_2, X_5, X_6. However, it is easy to note that the matrix is of rank 4, as should be. In fact, one should take into account the identities :

$$a_{Mi}(\theta_0) - a_{Ni}(\theta_0) = -\operatorname{ctg}\theta_0 a_{Qi}(\theta_0), \tag{3.89}$$

for $i = 1, 2, 5, 6$. Thus if we subtract the equations : $M_B(\theta_0) = 0$ and $N(\theta_0) = 0$, now independent of X_7 which is zero, we find the equation $Q(\pi/2) = 0$ multiplied by $(-\operatorname{ctg}\theta_0)$. Since $\theta_0 \neq \pi/2$ and $\theta_0 \neq 0$ we conclude that the first, second and third equations of (3.76) are linearly dependent. Therefore we arrive at four equations for the four unknowns : X_1, X_2, X_5, X_6

$$\mathbf{A}(\theta_0)\mathbf{X} = \mathbf{Q}, \quad \mathbf{Q} = [0,0,0,1]^T, \quad \mathbf{X} = [X_1, X_2, X_5, X_6]^T \tag{3.90}$$

with

$$\mathbf{A}(\theta_0) = \begin{bmatrix} a_{M1}(\theta_0) & a_{M2}(\theta_0) & a_{M5}(\theta_0) & a_{M6}(\theta_0) \\ a_{Q1}(\theta_0) & a_{Q2}(\theta_0) & a_{Q5}(\theta_0) & a_{Q6}(\theta_0) \\ a_{M1}(\pi/2) & a_{M2}(\pi/2) & a_{M1}(\pi/2) & a_{M2}(\pi/2) \\ a_{Q1}(\pi/2) & a_{Q2}(\pi/2) & -a_{Q1}(\pi/2) & -a_{Q2}(\pi/2) \end{bmatrix}, \tag{3.91}$$

where $a_{Mi}(\theta_0)$, $a_{Qi}(\theta_0)$ are given by (3.79), (3.88) and

$$\begin{aligned} a_{M1}(\pi/2) &= 2\rho^2 S_\sigma(\pi/2) + \nu R_\sigma(\pi/2), \\ a_{M2}(\pi/2) &= \nu S_\sigma(\pi/2) - 2\rho^2 R_\sigma(\pi/2), \\ a_{Q1}(\pi/2) &= 2\rho^2 S_\sigma^{-1}(\pi/2) + R_\sigma^{-1}(\pi/2), \\ a_{Q2}(\pi/2) &= S_\sigma^{-1}(\pi/2) - 2\rho^2 R_\sigma^{-1}(\pi/2). \end{aligned} \tag{3.92}$$

The displacement w_{θ_0} at $\theta = \pi/2$ is expressed by

$$w_{\theta_0}(\pi/2) = \frac{PR^3}{D}\check{w}_{\theta_0}(\pi/2)\ , \tag{3.93}$$

where

$$\check{w}_{\theta_0}(\pi/2) = -[(X_1 + X_5)R_\sigma(\pi/2) + (X_2 + X_6)S_\sigma(\pi/2)]. \tag{3.94}$$

The index θ_0 indicates that this quantity depends on θ_0. Note that according to (3.23)

$$f_2(\mathbf{u}_{\theta_0}) = \int_{\Gamma_\sigma} \widehat{Q} w ds = 2\pi R P w_{\theta_0}(\pi/2) \tag{3.95}$$

and by (3.21) $f_2(\mathbf{u}_{\theta_0}) = a_2(\mathbf{u}_{\theta_0}, \mathbf{u}_{\theta_0})$. The elastic potential reads

$$\mathcal{J}(\mathbf{v}) = \frac{1}{2} a_2(\mathbf{v}, \mathbf{v}) - f_2(\mathbf{v})$$

and its value on $\mathbf{v} = \mathbf{u}_{\theta_0}$ equals

$$\mathcal{J}(\mathbf{u}_{\theta_0}) = -\frac{1}{2} f_2(\mathbf{u}_{\theta_0}). \tag{3.96}$$

Thus

$$\mathcal{J}(\mathbf{u}_{\theta_0}) = -\frac{\pi P^2 R^4}{D} \check{w}_{\theta_0}(\pi/2) \tag{3.97}$$

or

$$\mathcal{J}(\mathbf{u}_{\theta_0}) = \frac{\pi P^2 R^4}{D} \check{\mathcal{J}}(\mathbf{u}_{\theta_0}) \tag{3.98}$$

and the non-dimensional potential equals

$$\check{\mathcal{J}}(\mathbf{u}_{\theta_0}) = R_\sigma(\pi/2)(X_1(\theta_0) + X_5(\theta_0)) + S_\sigma(\pi/2)(X_2(\theta_0) + X_6(\theta_0)). \tag{3.99}$$

The function $\check{\mathcal{J}}(\mathbf{u}_{\theta_0}) = -\check{w}_{\theta_0}(\pi/2)$ can be found explicitly and the rest of this section is devoted to this derivation.
Let us introduce new (complex) unknowns

$$\begin{array}{l} Z_1 = \frac{1}{2}(X_1 + iX_2), \quad Z_2 = \overline{Z}_1, \\ Z_5 = \frac{1}{2}(X_5 + iX_6), \quad Z_6 = \overline{Z}_5. \end{array} \tag{3.100}$$

The system (3.90) can be written in the form

$$\begin{array}{l} h_2(\theta_0) Z_2 + h_6(\theta_0) Z_6 = iD_1, \\ g_2(\theta_0) Z_2 - g_6(\theta_0) Z_6 = iD_2, \\ \beta(Z_2 + Z_6) = iD_3, \\ \alpha(Z_2 - Z_6) = \frac{1}{2} + iD_4 \end{array} \tag{3.101}$$

where $D_i \in \mathbb{R}$ are arbitrary constants,

$$\begin{aligned} h_2(\theta_0) &= \omega P_\sigma(\cos\theta_0) - (1-\nu)\operatorname{ctg}\theta_0 P^1_\sigma(\cos\theta_0),\\ h_6(\theta_0) &= \omega P_\sigma(-\cos\theta_0) + (1-\nu)\operatorname{ctg}\theta_0 P^1_\sigma(-\cos\theta_0),\\ g_2(\theta_0) &= \zeta P^1_\sigma(\cos\theta_0),\\ g_6(\theta_0) &= \zeta P^1_\sigma(-\cos\theta_0), \end{aligned} \tag{3.102}$$

and

$$\begin{aligned} &\zeta = 1 - i2\rho^2, \quad \omega = \nu - i2\rho^2,\\ &\alpha = \zeta P^1_\sigma(0), \quad \beta = \omega P_\sigma(0). \end{aligned} \tag{3.103}$$

Note that

$$\check{\mathcal{J}}(\mathbf{u}_{\theta_0}) = 2Re\{P_\sigma(0)(Z_2 + Z_6)\} \tag{3.104}$$

and by $(3.101)_3$

$$\check{\mathcal{J}}(\mathbf{u}_{\theta_0}) = 2Re\Big\{\frac{iD_3}{\omega}\Big\}$$

or

$$\check{\mathcal{J}}(\mathbf{u}_{\theta_0}) = 2Re\Big(\frac{i}{\nu - 2i\rho^2}\Big)D_3 = -\frac{4\rho^2}{|\omega|^2}D_3$$

and finally

$$\check{\mathcal{J}}(\mathbf{u}_{\theta_0}) = -\frac{4\rho^2}{|\omega|^2} Im(\beta(Z_2 + Z_6)). \tag{3.105}$$

To find D_3 one should solve the whole system (3.101). Let us re-write this system in the form

$$\mathbf{B}(\theta_0)\mathbf{Z} = \mathbf{Q}, \quad \mathbf{Z} = [Z_2, \overline{Z}_2, Z_6, \overline{Z}_6]^T \tag{3.106}$$

and

$$\mathbf{B}(\theta_0) = \begin{bmatrix} h_2(\theta_0) & \overline{h_2(\theta_0)} & h_6(\theta_0) & \overline{h_6(\theta_0)}\\ g_2(\theta_0) & \overline{g_2(\theta_0)} & -g_6(\theta_0) & -\overline{g_6(\theta_0)}\\ \beta & \overline{\beta} & \beta & \overline{\beta}\\ \alpha & \overline{\alpha} & -\alpha & -\overline{\alpha} \end{bmatrix} \tag{3.107}$$

The plan of the work is now the following. From the first and third equation of (3.107) we express Z_2 and Z_6 in terms of $\overline{Z}_2$ and $\overline{Z}_6$. Then we substitute these results into the second and fourth equation of (3.107). We find a 2×2

algebraic system for $\overline{Z}_2$ and $\overline{Z}_6$. We solve it and add $\overline{Z}_2 + \overline{Z}_6$. After heavy algebra we arrive at the following result

$$\beta(Z_2 + Z_6) = |\beta|^2 \frac{L(\theta_0)}{M(\theta_0)}, \tag{3.108}$$

where $|\beta|^2 = |\omega|^2 |P_\sigma(0)|^2$ and

$$\begin{array}{l} L = (\overline{h_2 - h_6})(g_2 + g_6) - (h_2 - h_6)(\overline{g_2 + g_6}), \\ M = \alpha\overline{\beta}[(g_6 - g_2)(\overline{h_2 - h_6}) + (\overline{g_6 + g_2})(h_2 + h_6)], \\ +\overline{\alpha}\beta[(\overline{g_6 - g_2})(h_2 - h_6) + (g_6 + g_2)(\overline{h_2 + h_6}) \\ -2\alpha\beta(\overline{g_2 h_6 + g_6 h_2}) - 2\overline{\alpha\beta}(g_2 h_6 + g_6 h_2), \end{array} \tag{3.109}$$

the arguments θ_0 being omitted for the sake of brevity.
Note that $Re(L) = 0$ and $Im(M) = 0$. By substituting the result (3.108) into (3.105) one finds

$$\check{\mathcal{J}}(\mathbf{u}_{\theta_0}) = -4\rho^2 |P_\sigma(0)|^2 \frac{L(\theta)}{iM(\theta)} \tag{3.110}$$

or

$$\check{\mathcal{J}}(\mathbf{u}_{\theta_0}) = -4\rho^2 |P_\sigma(0)|^2 F(\theta_0) \tag{3.111}$$

with

$$F(\theta_0) = \frac{Im\{(\overline{h_2 - h_6})(g_2 + g_6)\}}{Re\{\alpha\overline{\beta}[(h_2 + h_6)(\overline{g_2 + g_6}) - (\overline{h_2 - h_6})(g_2 - g_6)] - 2\alpha\beta(\overline{g_2 h_6 + g_6 h_2})\}}, \tag{3.112}$$

the argument θ_0 being suppressed for the sake of brevity.
The behaviour of $\check{\mathcal{J}}(\mathbf{u}_{\theta_0})$ in the vicinity of $\theta_0 = 0$ will be studied in Sec. 5.
Having found the constants Z_2, Z_6 we can determine X_i, $i = 1, 2, 5, 6$, or $A_{i(0)}$. To find the stress resultants $\widetilde{N}^{\alpha\beta}$ and moments $M^{\alpha\beta}$ one should use the formulae (3.28) and (3.31). The constants $A_{4(0)}$ and $A_{3(0)}$ will disappear and will not affect the formulae for $\widetilde{N}^{\alpha\beta}$ and $M^{\alpha\beta}$. To find the state of displacements (not only w, but also u and v) one should take into account the boundary conditions of kinematic type.

4. Topological derivative of the total compliance of a spherical shell

4.1. The notion of topological derivative

The topological derivative has been introduced in the paper by Sokolowski and Żochowski (1997). Let us recall its definition for a functional defined on S. Functionals defined by a mapping

$$\mathcal{J} : S \setminus K \mapsto \mathbb{R} \tag{4.1}$$

for any compact set $K \subset \overline{S}$ are called shape functionals. Let us consider a family of neighbourhoods $B_\rho(A)$, $A \in S$, $\rho > 0$ determined by equidistance contours on S. Assume that the following limit exists

$$\mathcal{F}(A) = \lim_{\rho \searrow 0} \frac{\mathcal{J}(S \setminus \overline{B_\rho(A)}) - \mathcal{J}(S)}{|\overline{B_\rho(A)}|}. \tag{4.2}$$

The function $\mathcal{F}(A)$, $A \in S$ is called a topological derivative of $\mathcal{J}(S)$.
For the case of a spherical shell the neighbourhoods B_ρ have circular boundaries of radii $\rho = R \sin\theta_0$, see Fig. 4.1

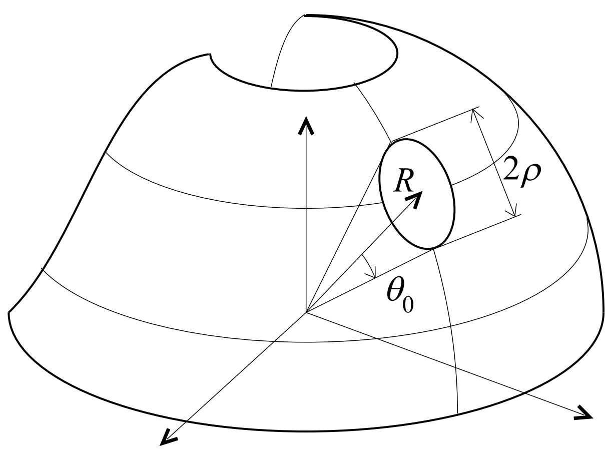

Figure 4: *Fig. 4.1. Opening in a spherical shell*

The area of B_ρ equals

$$|B_\rho| = 2(1 - \cos\theta_0)\pi R^2 \tag{4.3}$$

or

$$|B_\rho| = 4\sin^2\frac{\theta_0}{2}\pi R^2. \tag{4.4}$$

Let us define the function

$$\mathcal{J}_1(\rho) = \mathcal{J}(S \setminus \overline{B_\rho(A)}) \tag{4.5}$$

for $\rho \geq 0$. The main conjecture of the paper is that $\mathcal{J}_1(\rho)$ can be expanded as follows

$$\mathcal{J}_1(\rho) = \mathcal{J}(S) + \frac{1}{2}\rho^2\mathcal{J}_1''(0_+) + o(\rho^2) \tag{4.6}$$

and $|\mathcal{J}_1''(0_+)| < +\infty$. Since $\rho = R\sin\theta_0$ it will be more convenient to consider the function $j(\theta_0) = \mathcal{J}_1(R\sin\theta_0)$ and look for the expansion

$$j(\theta_0) = \mathcal{J}(S) + \frac{1}{2}(\theta_0)^2 j''(0_+) + o((\theta_0)^2). \tag{4.7}$$

Here $(\ )' = d/d\theta_0$. Before proceeding further we have to rearrange the problem $(\tilde{\mathrm{P}})$ of Sec. 3 such that it involves physical components of all mechanical entities.

4.2. Setting of the problem $(\tilde{\mathbf{P}})$ in physical components

In the tensorial notation, see e.g. Eq. (3.18), the variables θ and ϕ are concealed, which is highly inconvenient in solving the optimization problems. We have to use a standard notation, namely u, v, w are displacements (see Sec. 3.3), the membrane deformations are given by (3.46)-(3.47).
Now we introduce the physical components of the change of curvature tensor $(\tilde{\rho}_{\alpha\beta})$

$$\kappa_{\alpha\beta} = \frac{\tilde{\rho}_{\alpha\beta}}{\sqrt{a_{\alpha\alpha}}\sqrt{a_{\beta\beta}}}\ . \tag{4.8}$$

Hence we find

$$\begin{aligned} \kappa_{11} &= \tfrac{1}{R^2}\Big(\tfrac{\partial^2 w}{\partial\theta^2} + w\Big), \\ \kappa_{22} &= \tfrac{1}{R^2\sin^2\theta}\Big(\tfrac{\partial^2 w}{\partial\phi^2} + \sin\theta\cos\theta\tfrac{\partial w}{\partial\theta} + \sin^2\theta\ w\Big), \\ \kappa_{12} &= \tfrac{1}{R^2\sin\theta}\Big(\tfrac{\partial^2 w}{\partial\theta\partial\phi} - \mathrm{ctg}\,\theta\tfrac{\partial w}{\partial\phi}\Big). \end{aligned} \tag{4.9}$$

Physical components of $\widetilde{N}^{\alpha\beta}$ are denoted by $N^{\alpha\beta}$ and physical components of $M^{\alpha\beta}$ are denoted by the same letter M, or $M^{\alpha\beta}$.
The constitutive relations (3.17) assume the form

$$\begin{array}{l} N^{11} = C(\varepsilon_{11} + \nu\varepsilon_{22}), \quad N^{22} = C(\varepsilon_{22} + \nu\varepsilon_{11}), \\ N^{12} = N^{21} = C(1-\nu)\varepsilon_{12} \; ; \\ M^{11} = D(\kappa_{11} + \nu\kappa_{22}), \quad M^{22} = D(\kappa_{22} + \nu\kappa_{11}), \\ M^{12} = M^{21} = D(1-\nu)\kappa_{12} \end{array} \tag{4.10}$$

and $C = Eh(1-\nu^2)^{-1}$. Consider a spherical shell, free of surface loading p^i, parametrized by $\xi^1 = \theta$, $\xi^2 = \phi$, of two edges :

$$\theta = \theta_0 \;\; \text{(the upper edge } \Gamma_0)$$

and

$$\theta = \theta_1(\phi) \;\; \text{the lower edge } \Gamma_1)$$

and $\phi \in [0, 2\pi]$, see Fig. 4.2

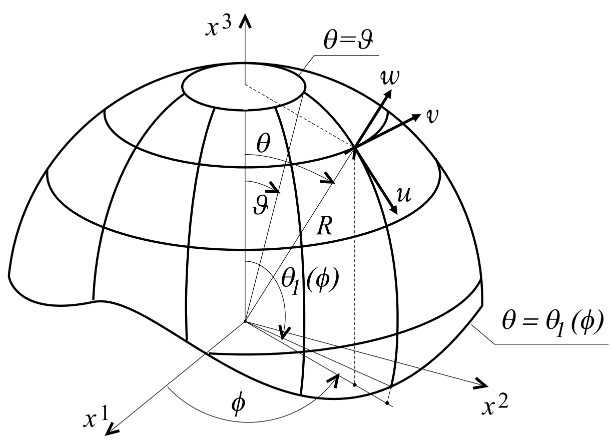

Figure 5: *Fig. 4.2. Opening located at a north-pole*

The equations of the edges are
$\mathbf{\Phi} = R[\sin\theta_1(\phi)\cos\phi, \sin\theta_1(\phi)\sin\phi, \cos\theta_1(\phi)]$
- the lower edge and

$\mathbf{\Phi} = R[\sin\theta_0\cos\phi, \sin\theta_0\sin\phi, \cos\theta_0]$
-the upper edge.

Assume that the lower edge is subjected to a self-equilibrated system of boundary forces $\widehat{N}$, $\widehat{S}$, $\widehat{Q}$ and $\widehat{M}_B$ and the linear form f_2 reads

$$f_2(\overline{u},\overline{v},\overline{w}) = \int_{\Gamma_1}\Big(\widehat{N}v_n + \widehat{S}v_t + \widehat{Q}\overline{w} + \widehat{M}_B\frac{\partial\overline{\mathbf{w}}}{\partial n}\Big)ds \tag{4.11}$$

where $\mathbf{v} = (\overline{u},\overline{v},\overline{w})$ is the vector of trial displacements and $v_n = \mathbf{v}.\mathbf{n}$, $v_t = \mathbf{v}.\mathbf{t}$. The density of the form $a_2(.,.)$ reads

$$\begin{aligned} & e(u,v,w\,;\overline{u},\overline{v},\overline{w}) = C[\varepsilon_{11}(u,w)\varepsilon_{11}(\overline{u},\overline{w})+ \\ & +\nu[\varepsilon_{11}(u,w)\varepsilon_{22}(\overline{u},\overline{v},\overline{w}) + \varepsilon_{22}(u,v,w)\varepsilon_{11}(\overline{u},\overline{w})]+ \\ & +\varepsilon_{22}(u,v,w)\varepsilon_{22}(\overline{u},\overline{v},\overline{w})]+ \\ & +2(1-\nu)\varepsilon_{12}(u,v)\varepsilon_{12}(\overline{u},\overline{v})]+ \\ & +D[\kappa_{11}(w)\kappa_{11}(\overline{w})+ \\ & +\nu[\kappa_{11}(w)\kappa_{22}(\overline{w}) + \kappa_{22}(w)\kappa_{11}(\overline{w})]+ \\ & +\kappa_{22}(w)\kappa_{22}(\overline{w}) + 2(1-\nu)\kappa_{12}(w)\kappa_{12}(\overline{w})]. \end{aligned} \tag{4.12}$$

Now we define the bilinear forms $a_2(.,.) = a_{\theta_0}(.,.)$ in terms of physical components

$$a_{\theta_0}(u,v,w\,;\overline{u},\overline{v},\overline{w}) = \int_0^{2\pi}\int_{\theta_0}^{\theta_1(\phi)} e(u,v,w\,;\overline{u},\overline{v},\overline{w})R^2\sin\theta d\theta d\phi. \tag{4.13}$$

The variational formulation of the equilibrium problem is as follows

$$(\mathrm{P}_{\theta_0})\left\{\begin{array}{l}\text{find}\ \ (u_{\theta_0},v_{\theta_0},w_{\theta_0})\in V_{\theta_0}\ \ \text{such that} \\ a_{\theta_0}(u_{\theta_0},v_{\theta_0},w_{\theta_0}\,;\overline{u},\overline{v},\overline{w}) = f_2(\overline{u},\overline{v},\overline{w}) \quad \forall(\overline{u},\overline{v},\overline{w})\in V_{\theta_0}\end{array}\right.$$

where $V_{\theta_0} = (H^1(\mathcal{O}_{\theta_0}))^2 \times H^2(\mathcal{O}_{\theta_0})$, the domain $\mathcal{O}_{\theta_0}$ refers to the shell of Fig. 4.2 and $f_2(.)$ satisfies

$$f_2(\overline{u},\overline{v},\overline{w}) = 0 \quad \forall(\overline{u},\overline{v},\overline{w})\in\mathcal{R}.$$

The variational equilibrium equation reads

$$\begin{aligned} & \textstyle\int_0^{2\pi}\int_{\theta_0}^{\theta_1(\phi)}[N_{\theta_0}^{\alpha\beta}\varepsilon_{\alpha\beta}(\overline{u},\overline{v},\overline{w}) + M_{\theta_0}^{\alpha\beta}\kappa_{\alpha\beta}(\overline{w})]R^2\sin\theta d\theta d\phi \\ = \ & f_2(\overline{u},\overline{v},\overline{w}) \quad \forall(\overline{u},\overline{v},\overline{w})\in V_{\theta_0}\ , \end{aligned} \tag{4.14}$$

where $N_{\theta_0}^{\alpha\beta}$ and $M_{\theta_0}^{\alpha\beta}$ are physical components of the stress and couple resultants. The index θ_0 indicates that the solution depends on this quantity. The middle surface of the shell is denoted by S_{θ_0}.

4.3. Computing the topological derivative of the total compliance

Let us define the shape functional

$$\mathcal{J}(S_{\theta_0}) = \mathcal{J}(\mathbf{u}_{\theta_0}) \tag{4.15}$$

and let $j(\theta_0) = \mathcal{J}(\mathbf{u}_{\theta_0})$, where $\mathbf{u}_{\theta_0} = (u_{\theta_0}, v_{\theta_0}, w_{\theta_0})$ and $\mathcal{J}$ is defined by (3.96), i.e. it is a total compliance of the shell, taken with the negative sign. Our aim is to find its first and second derivatives with respect to θ_0. By the principle of minimum of potential energy

$$\mathcal{J}(\mathbf{u}_{\theta_0}) = \min_{\overline{\mathbf{u}}} \mathcal{J}(\overline{\mathbf{u}}) = \min_{\overline{\mathbf{u}}} \Big\{ \frac{1}{2} a_{\theta_0}(\overline{\mathbf{u}}, \overline{\mathbf{u}}) - f_2(\overline{\mathbf{u}}) \Big\}. \tag{4.16}$$

One can prove that

$$\frac{dj(\theta_0)}{d\theta_0} = \frac{1}{2} a'_{\theta_0}(\mathbf{u}_{\theta_0}, \mathbf{u}_{\theta_0}), \tag{4.17}$$

and $a'_{\theta_0}(.,.) = \dfrac{\partial a_{\theta_0}(.,.)}{\partial \theta_0}$ is the partial derivative with respect to θ_0. We compute

$$a'_{\theta_0}(\mathbf{u}_{\theta_0}, \mathbf{u}_{\theta_0}) = \frac{\partial}{\partial \theta_0} \int_0^{2\pi} \int_{\theta_0}^{\theta_1(\phi)} e(\mathbf{u}_{\theta_0}, \mathbf{u}_{\theta_0}) R^2 \sin\theta d\theta d\phi \tag{4.18}$$

and find

$$a'_{\theta_0}(\mathbf{u}_{\theta_0}, \mathbf{u}_{\theta_0}) = -R^2 \sin\theta_0 \int_0^{2\pi} e(\mathbf{u}_{\theta_0}, \mathbf{u}_{\theta_0})(\theta_0, \phi) d\phi. \tag{4.19}$$

The integrand $e(\mathbf{u}_{\theta_0}, \mathbf{u}_{\theta_0})(\theta_0, \phi)$ represents the elastic energy measured along the Γ_0 edge. But the variational equation of equilibrium (4.14) implies

$$N_{\theta_0}^{11} = 0, \quad N_{\theta_0}^{12} = 0, \quad M_{\theta_0}^{11} = 0 \tag{4.20}$$

which means that $e(\mathbf{u}_{\theta_0}, \mathbf{u}_{\theta_0})(\theta_0, \phi)$ can be expressed in terms of $N_{\theta_0}^{22}$, $M_{\theta_0}^{22}$, $M_{\theta_0}^{12}$, solely, by using the constitutive relations inverse to (4.10). From the literature on special solutions of shells with openings (see e.g. Grigoliuk and Filshtinskii (1970)) we know that the stress and couple resultants mentioned above are of the same order as the loading applied to the lower edge Γ_1. One can say that the loading at Γ_1 is transmitted into the circumference of the opening in the form of so-called hoop stresses, but they do not grow if the dimension of the opening tends to zero. This means that $a'_{\theta_0}(\mathbf{u}_{\theta_0}, \mathbf{u}_{\theta_0})$ tends to zero. We conclude that the first derivative of $j(\theta_0)$ vanishes.

Remark 4.1. In the case of an infinite plate with an opening, loaded in the infinity by in-plane loadings, the hoop stresses are completely independent of the dimensions of the openings. This must be the case since the plane-stress (or membrane) theory is free from any internal length scales, hence one cannot distinguish between small and large openings if the plate is unbounded and is weakened by only one opening.
The same remark concerns the Kirchhoff theory of bending. The thin shell theory used here couples both the membrane and bending effects weakly, since the constitutive relations (4.10) are uncoupled. Along the openings no new effects should occur. □

Let us proceed to deriving a formula for the second derivative of the function $j(\theta_0)$.
By (4.17) and (4.19) we find

$$2j''(\theta_0) = -R^2 \cos\theta_0 \int_0^{2\pi} e(\mathbf{u}_{\theta_0}, \mathbf{u}_{\theta_0})(\theta_0, \phi)d\phi - 2R^2 \sin\theta_0 \int_0^{2\pi} e(\mathbf{u}_{\theta_0}, \mathbf{u}'_{\theta_0})(\theta_0, \phi)d\phi, \tag{4.21}$$

where $j''(\theta_0) = d^2 j(\theta_0)/d(\theta_0)^2$. Thus $j''(\theta_0)$ depends on a new field $\mathbf{u}'_{\theta_0}$. Differentiation of the both sides of the equation of the (P_{θ_0}) problem gives

$$(\mathrm{P}'_{\theta_0}) \begin{cases} \text{find } \mathbf{u}'_{\theta_0} = (u'_{\theta_0}, v'_{\theta_0}, w'_{\theta_0}) \in V_{\theta_0} \text{ such that} \\ a_{\theta_0}(\mathbf{u}'_{\theta_0}, \mathbf{v}) = -a'_{\theta_0}(\mathbf{u}_{\theta_0}, \mathbf{v}) \quad \forall \mathbf{v} \in V_{\theta_0} \end{cases}$$

where $\mathbf{v} = (\overline{u}, \overline{v}, \overline{w})$ and

$$a'_{\theta_0}(\mathbf{u}_{\theta_0}, \mathbf{v}) = -R^2 \sin\theta_0 \int_0^{2\pi} e(\mathbf{u}_{\theta_0}, \mathbf{v})(\theta_0, \phi)d\phi \tag{4.22}$$

or, by (4.10) we have

$$a'_{\theta_0}(\mathbf{u}_{\theta_0}, \mathbf{v}) = -R^2 \sin\theta_0 \int_0^{2\pi} [N^{\alpha\beta}_{\theta_0} \varepsilon_{\alpha\beta}(\overline{u}, \overline{v}, \overline{w}) + M^{\alpha\beta}_{\theta_0} \kappa_{\alpha\beta}(\overline{w})](\theta_0, \phi)d\phi. \tag{4.23a}$$

Taking into account the boundary conditions (4.20) one can reduce the expression above to the form

$$\begin{aligned} a'_{\theta_0}(\mathbf{u}_{\theta_0}, \mathbf{v}) = -R^2 \sin\theta_0 \textstyle\int_0^{2\pi} [N^{22}_{\theta_0} \varepsilon_{22}(\overline{u}, \overline{v}, \overline{w}) + \\ + M^{22}_{\theta_0} \kappa_{22}(\overline{w}) + 2M^{12}_{\theta_0} \kappa_{12}(\overline{w})](\theta_0, \phi) d\phi. \end{aligned} \tag{4.23b}$$

Upon substitution of $(3.47)_2$ and (4.9) one finds

$$\begin{aligned} a'_{\theta_0}(\mathbf{u}_{\theta_0}, \mathbf{v}) = -R^2 \sin\theta_0 \textstyle\int_0^{2\pi} \Big[\frac{1}{R\sin\theta_0} N^{22}_{\theta_0} \Big(\frac{\partial \overline{v}}{\partial \phi} + \cos\theta_0 \ \overline{u} + \sin\theta_0 \ \overline{w} \Big) + \\ + \frac{2}{R^2 \sin\theta_0} M^{12}_{\theta_0} \Big(\frac{\partial^2 \overline{w}}{\partial\theta \partial\phi} - \operatorname{ctg}\theta_0 \frac{\partial \overline{w}}{\partial \phi} \Big) + \frac{1}{R^2 \sin^2\theta_0} M^{22}_{\theta_0} \times \\ \times \Big(\frac{\partial^2 \overline{w}}{\partial \phi^2} + \sin\theta_0 \cos\theta_0 \frac{\partial \overline{w}}{\partial \theta} + \sin^2\theta_0 \ \overline{w} \Big)(\theta_0, \phi) \Big] d\phi. \end{aligned} \tag{4.24}$$

The next step requires imposing further continuity restrictions on $N^{22}_{\theta_0}$, $M^{\alpha 2}_{\theta_0}$ and $\partial\overline{w}/\partial\theta$ along $\Gamma_0 : N^{22}_{\theta_0} \in C^1(\Gamma_0)$, $M^{12}_{\theta_0} \in C^1(\Gamma_0)$, $M^{22}_{\theta_0} \in C^2(\Gamma_0)$, $\partial\overline{w}/\partial\theta \in C(\Gamma_0)$. Using the identities

$$\begin{aligned} N^{22}_{\theta_0} \frac{\partial \overline{v}}{\partial \phi} &= \underline{\frac{\partial}{\partial \phi}(N^{22}_{\theta_0} \overline{v})} - \frac{\partial N^{22}_{\theta_0}}{\partial \phi} \overline{v}, \\ M^{12}_{\theta_0} \frac{\partial^2 \overline{w}}{\partial\theta\partial\phi} &= \underline{\frac{\partial}{\partial \phi}\Big(M^{12}_{\theta_0} \frac{\partial \overline{w}}{\partial \theta}\Big))} - \frac{\partial M^{12}_{\theta_0}}{\partial \phi} \frac{\partial \overline{w}}{\partial \theta}, \\ M^{12}_{\theta_0} \frac{\partial \overline{w}}{\partial \phi} &= \underline{\frac{\partial}{\partial \phi}(M^{12}_{\theta_0} \overline{w})} - \frac{\partial M^{12}_{\theta_0}}{\partial \phi} \overline{w}, \\ M^{22}_{\theta_0} \frac{\partial^2 \overline{w}}{\partial \phi^2} &= \underline{\frac{\partial}{\partial \phi}\Big(M^{22}_{\theta_0} \frac{\partial \overline{w}}{\partial \phi} - \frac{\partial M^{22}_{\theta_0}}{\partial \phi} \overline{w}\Big)} + \frac{\partial^2 M^{22}_{\theta_0}}{\partial \phi^2} \overline{w} \end{aligned}$$

and noting that the integrals of the underscored terms vanish one rearranges the expression (4.24) to the form

$$\begin{aligned} a'_{\theta_0}(\mathbf{u}_{\theta_0}, \mathbf{v}) = -\textstyle\int_{\Gamma_0} \Big\{ (N^{22}_{\theta_0} \operatorname{ctg}\theta_0) \overline{u}(\theta_0, \phi) + \Big(-\frac{1}{\sin\theta_0} \frac{\partial N^{22}_{\theta_0}}{\partial \phi} \Big) \overline{v}(\theta_0, \phi) + \\ + \Big(N^{22}_{\theta_0} + \frac{2\cos\theta_0}{R\sin^2\theta_0} \frac{\partial M^{12}_{\theta_0}}{\partial \phi} + \frac{1}{R\sin^2\theta_0} \frac{\partial^2 M^{22}_{\theta_0}}{\partial \phi^2} + \frac{1}{R} M^{22}_{\theta_0} \Big) \overline{w}(\theta_0, \phi) + \\ + \Big(-\frac{2}{\sin\theta_0} \frac{\partial M^{12}_{\theta_0}}{\partial \phi} + \operatorname{ctg}\theta_0 M^{22}_{\theta_0} \Big) \frac{\partial \overline{w}}{\partial n}(\theta_0, \phi) \Big\} ds, \end{aligned} \tag{4.25}$$

where $ds = R\sin\theta_0 d\phi$.

Now we are ready to find the strong formulation of (P'_{θ_0}). The local equilibrium equations are homogeneous ; they have the form (3.26) if one neglects p^i and express them in terms of physical components. The boundary conditions on

Γ_1 are also homogeneous ; this boundary is unloaded. The boundary Γ_0 is loaded and there the boundary conditions are of the form

$$\begin{aligned} N(\mathbf{u}'_{\theta_0}) &= \operatorname{ctg}\theta_0 N^{22}_{\theta_0}, \\ S(\mathbf{u}'_{\theta_0}) &= -\tfrac{1}{\sin\theta_0}\tfrac{\partial N^{22}_{\theta_0}}{\partial\phi}, \\ Q(\mathbf{u}'_{\theta_0}) &= N^{22}_{\theta_0} + \tfrac{1}{R}M^{22}_{\theta_0} + \tfrac{2\cos\theta_0}{\sin^2\theta_0}\tfrac{\partial M^{12}_{\theta_0}}{\partial\phi} + \tfrac{1}{R\sin^2\theta_0}\tfrac{\partial^2 M^{22}_{\theta_0}}{\partial\phi^2}, \\ M_B(\mathbf{u}'_{\theta_0}) &= \operatorname{ctg}\theta_0 M^{22}_{\theta_0} - \tfrac{2}{\sin\theta_0}\tfrac{\partial M^{12}_{\theta_0}}{\partial\phi} \end{aligned} \tag{4.26}$$

and here $N = N^{11}$, $S = N^{12}$, $M_B = M^{11}$.

We note that Q has higher singularity than N, S and M_B. This is due to the fact that the transverse shear forces are, roughly speaking, derivatives of moments. Thus the $(\sin\theta_0)^{-2}$ singularity in Q has the same effect as $(\sin\theta_0)^{-1}$ singularity of M_B.

The singularities of N, S, Q and M_B given along Γ_0 transmit into the generalized hoop stresses $N^{22}(\mathbf{u}'_{\theta_0})$, $M^{22}(\mathbf{u}'_{\theta_0})$, $M^{12}(\mathbf{u}'_{\theta_0})$. Now we are ready to estimate the second term of (4.21). The expression $e(\mathbf{u}_{\theta_0}, \mathbf{u}'_{\theta_0})$ can be rearranged to the form of a sum of the product of the quantities $N^{22}_{\theta_0}$, $M^{22}_{\theta_0}$, $M^{12}_{\theta_0}$, associated with $\mathbf{u}_{\theta_0}$, and the quantities $N^{22}(\mathbf{u}'_{\theta_0})$, $M^{22}(\mathbf{u}'_{\theta_0})$, $M^{12}(\mathbf{u}'_{\theta_0})$. The former ones are of order of the loading applied to Γ_1. The latter ones are of order $(\sin\theta_0)^{-1}$. The result is that this singularity is cancelled with the factor $(\sin\theta_0)$ and we conclude that both the terms of (4.21 are of the same order.

Let us analyse now the quantity

$$g(\theta_0) = \int_0^{2\pi} e(\mathbf{u}_{\theta_0}, \mathbf{u}_{\theta_0})(\theta_0, \phi)d\phi. \tag{4.27}$$

Due to the opening Γ_0 being unloaded we have (4.20). The integrand of (4.27) depends on $(N^{22}_{\theta_0})^2$, $(M^{22}_{\theta_0})^2$, $(M^{12}_{\theta_0})^2$ at (θ_0, ϕ) points. If $\theta_0 \searrow 0$ these quantities can be expressed in terms of the principal values of $N_0^{\alpha\beta}$ and $M_0^{\alpha\beta}$, associated with the problem of a shell without the opening Γ_0. For the plane stress problem ($M^{\alpha\beta} = 0$) such formulae are reported in Sokolowski and Żochowski (1997) and for the pure bending problem ($N^{\alpha\beta} = 0$) case are given by Schumacher (1996). Thus we anticipate here that similar formulae hold in the shell case. Since the reciprocal terms (coupling $\mathbf{N}$ and $\mathbf{M}$) are absent in (4.27) we should get

$$\lim_{\theta_0 \searrow 0} g(\theta_0) = 2\pi \left[\frac{1}{Eh}(\gamma_1 (tr\mathbf{N}_0)^2 + \gamma_2 tr(\mathbf{N}_0)^2) + \frac{1}{Eh^3}(\delta_1 (tr\mathbf{M}_0)^2 + \delta_2 tr(\mathbf{M}_0)^2) \right] \tag{4.28}$$

with γ_α, δ_α being constants depending on the Poisson ratio ν. This yields

$$2j''(\theta_0) = -\frac{2\pi R^2}{E}\Big\{\frac{1}{h}(\gamma_1(tr\mathbf{N}_0)^2+\gamma_2 tr(\mathbf{N}_0)^2)+\frac{1}{h^3}(\delta_1(tr\mathbf{M}_0)^2+\delta_2 tr(\mathbf{M}_0)^2)\Big\}-\Delta\,, \tag{4.29}$$

where

$$\Delta = 2R^2 \lim_{\theta_0 \searrow 0} \int_0^{2\pi} e(\mathbf{u}_{\theta_0}, \sin\theta_0 \mathbf{u}'_{\theta_0})(\theta_0, \phi)d\phi. \tag{4.30}$$

The first term in (4.29) is constructive, since the coefficients γ_α, δ_α can be found by analysing special cases of loading. Let us stress once again that they are material constants. In contrary, the formula for Δ is less constructive. In the case of plane-stress ($M^{\alpha\beta} = 0$) $\Delta = 0$, which has been proved by Sokolowski and Żochowski (1997). In the pure bending case ($N^{\alpha\beta} = 0$) this formula is absent in the paper by Schumacher (1996). These facts suggest that $\Delta = 0$ also for shells, but the proof is unknown.

5. Particular case : finding the topological derivative of the compliance functional at the north-pole of a hemisphere subjected to a constant circumferential transverse loading

5.1. The aim of the present section is to analyse the behaviour of the functional $\check{\mathcal{J}}(\mathbf{u}_{\theta_0})$ given by (3.99) for small values of θ_0 for the spherical shell considered in Sec. 3.4. First we show that

$$(X_1(\theta_0), X_2(\theta_0)) \to (X_1(0), X_2(0)) \text{ and } (X_5(\theta_0), X_6(\theta_0)) \to (0,0)$$

if $\theta_0 \searrow 0$, where $X_1(0)$, $X_2(0)$ correspond to the case of a hemispherical shell closed at a north-pole. To prove it we analyse the solution of the algebraic system (3.90) with the matrix $\mathbf{A}(\theta_0)$ given by (3.91). According to the remarks in the Appendix the coefficients $a_{M5}(\theta_0)$, $a_{M6}(\theta_0)$, $a_{Q5}(\theta_0)$ and $a_{Q6}(\theta_0)$ are singular at $\theta_0 = 0$. Thus we have to regularize this system before analysing the behaviour of $X_i(\theta_0)$ in the neighbourhood of $\theta_0 = 0$. To get rid of the singularities in the two first equations we multiply the first one by $\sin^2\theta_0$ and the second one by $\sin\theta_0$. Then the terms $\operatorname{ctg}\theta_0 R_\sigma^{-1}(\theta_0)$, $\operatorname{ctg}\theta_0 S_\sigma^{-1}(\theta_0)$ and $\operatorname{ctg}\theta_0 R_\sigma^{-1}(\pi-\theta_0)$, $\operatorname{ctg}\theta_0 S_\sigma^{-1}(\pi-\theta_0)$ will be replaced with

$$\sin\theta_0 R_\sigma^{-1}(\theta_0), \quad \sin\theta_0 S_\sigma^{-1}(\theta_0) \tag{5.1}$$

and with

$$\sin\theta_0 R_\sigma^{-1}(\pi-\theta_0), \quad \sin\theta_0 S_\sigma^{-1}(\pi-\theta_0), \tag{5.2}$$

respectively. This new algebraic system has the form

$$\widetilde{\mathbf{A}}(\theta_0)\mathbf{X}(\theta_0) = \mathbf{Q}, \tag{5.3}$$

where the matrix $\widetilde{\mathbf{A}}(\theta_0)$ is formed from the matrix $\mathbf{A}(\theta_0)$ by multiplying its two first raws by $\sin^2\theta_0$ and $\sin\theta_0$, respectively :

$$\widetilde{\mathbf{A}}(\theta_0) = \begin{bmatrix} \tilde{a}_{M1}(\theta_0) & \tilde{a}_{M2}(\theta_0) & \tilde{a}_{M5}(\theta_0) & \tilde{a}_{M6}(\theta_0) \\ \tilde{a}_{Q1}(\theta_0) & \tilde{a}_{Q2}(\theta_0) & \tilde{a}_{Q5}(\theta_0) & \tilde{a}_{Q6}(\theta_0) \\ a_{M1}(\pi/2) & a_{M2}(\pi/2) & a_{M1}(\pi/2) & a_{M2}(\pi/2) \\ a_{Q1}(\pi/2) & a_{Q2}(\pi/2) & -a_{Q1}(\pi/2) & -a_{Q2}(\pi/2), \end{bmatrix} \tag{5.4}$$

with

$$\tilde{a}_{Mi}(\theta_0) = \sin^2\theta_0 a_{Mi}(\theta_0), \quad \tilde{a}_{Qi}(\theta_0) = \sin\theta_0 a_{Qi}(\theta_0), \quad i \in \{1,2,5,6\}. \tag{5.5}$$

Since the expressions (5.1) tend to zero and the limits of the expressions (5.2) are finite (see A.10) the system (5.3) degenerates to

$$\begin{bmatrix} 0 & 0 & c_1 & c_2 \\ 0 & 0 & c_3 & c_4 \\ a_{M1}(\pi/2) & a_{M2}(\pi/2) & a_{M1}(\pi/2) & a_{M2}(\pi/2) \\ a_{Q1}(\pi/2) & a_{Q2}(\pi/2) & -a_{Q1}(\pi/2) & -a_{Q2}(\pi/2) \end{bmatrix} \begin{bmatrix} X_1(0) \\ X_2(0) \\ X_5(0) \\ X_6(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{5.6}$$

where

$$\begin{aligned} c_1 &= \lim_{\theta_0 \searrow 0} (\sin^2\theta_0 a_{M5}(\theta_0)), \\ c_2 &= \lim_{\theta_0 \searrow 0} (\sin^2\theta_0 a_{M6}(\theta_0)), \\ c_3 &= \lim_{\theta_0 \searrow 0} (\sin\theta_0 a_{Q5}(\theta_0)), \\ c_4 &= \lim_{\theta_0 \searrow 0} (\sin\theta_0 a_{Q6}(\theta_0)). \end{aligned} \tag{5.7}$$

Let us prove that the matrix

$$\mathbf{a}_{56} = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} \tag{5.8}$$

is nonsingular. We compute these numbers with using (A.10) and the definitions (3.79) and (3.88) :

$$\begin{aligned} c_1 &= -\tfrac{2}{\pi}(1-\nu)\sin(\pi a)\cosh(\pi b), \\ c_2 &= -\tfrac{2}{\pi}(1-\nu)\cos(\pi a)\sinh(\pi b), \\ c_3 &= \tfrac{2}{\pi}[\sin(\pi a)\cosh(\pi b) + 2\rho^2\cos(\pi a)\sinh(\pi b)], \\ c_4 &= \tfrac{2}{\pi}[\cos(\pi a)\sinh(\pi b) - 2\rho^2\sin(\pi a)\cosh(\pi b)], \end{aligned} \tag{5.9}$$

where $a + ib = \sigma$, see (3.62). After some algebra we find

$$\det \mathbf{A}_{56} = 2\rho^2|\sin(\pi\sigma)|^2 > 0 \tag{5.10}$$

if $\sigma \notin \mathbb{N}$. But σ is always a complex number. Thus $\det \mathbf{A}_{56} \neq 0$.
Now we come back to the system (5.6) and conclude that the constants $X_1(0)$ and $X_2(0)$ can be found by solving the system

$$\begin{bmatrix} a_{M1}(\pi/2) & a_{M2}(\pi/2) \\ a_{Q1}(\pi/2) & a_{Q2}(\pi/2) \end{bmatrix} \begin{bmatrix} X_1(0) \\ X_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{5.11}$$

and

$$X_5(0) = X_6(0) = 0 \tag{5.12}$$

One can easy note that the system (5.11) corresponds to the problem of a hemisphere (closed at the north-pole) subjected to $\widehat{Q} = P$ along the lower edge, see Fig. 5.1. Indeed,

in this case the solution $\hat{w}_0(\theta)$ and $\widehat{\Phi}_0(\theta)$ is sought in the form

$$\begin{aligned} &\hat{w}_0(\theta) = -\tfrac{1-\nu^2}{kR^2}[\widehat{A}_{1(0)}R_\sigma(\theta) + \widehat{A}_{2(0)}S_\sigma(\theta) + \widehat{A}_{4(0)}\cos\theta] \ , \\ &\widehat{\Phi}_0(\theta) = \tfrac{1}{R^2}[(\nu-1)\widehat{A}_{3(0)}\cos\theta + (\nu\widehat{A}_{1(0)} - 2\rho^2\widehat{A}_{2(0)})R_\sigma(\theta) + \\ &+ (\nu\widehat{A}_{2(0)} + 2\rho^2\widehat{A}_{1(0)})S_\sigma(\theta)] \end{aligned} \tag{5.13}$$

and the conditions

$$M_B(\pi/2) = 0, \ \ Q(\pi/2) = P \tag{5.14}$$

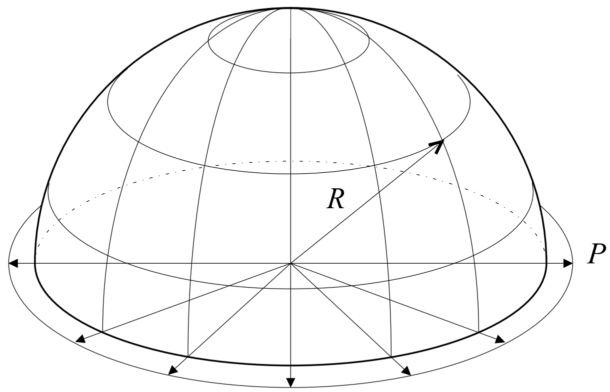

Figure 6: *Fig. 5.1. A hemispherical shell subjected to a transverse loading*

give the system (5.11). The equation $N(\pi/2) = 0$ is identical with $M_B(\pi/2) = 0$. Thus the result (5.11), (5.12) discloses a correspondence between the solution to the problem with an opening. Now we know that if $\theta_0 \searrow 0$ the constants X_1, X_2 tend to $X_1(0)$, $X_2(0)$ characterizing the solution for a shell without an opening at the north-pole and the constants X_5, X_6 tend to zero.

5.2. Now we prove that

$$\frac{d\check{\mathcal{J}}(\mathbf{u}_{\theta_0})}{d\theta_0}(0) = 0 \tag{5.15}$$

and to this end we make use of the formula (3.110). Both the functions L and M are multiplied by $\sin^3\theta_0$ to suppress the singularities. We find

$$\check{\mathcal{J}}(\mathbf{u}_{\theta_0}) = -4\rho^2|P_\sigma(0)|^2\frac{\widetilde{L}(\theta_0)}{i\widetilde{M}(\theta_0)} \tag{5.16}$$

with $\widetilde{L}$ and $\widetilde{M}$ depending on the functions

$$\begin{aligned}
\hat{h}_2(\theta) &= \sin^2\theta\ \omega P_\sigma(\cos\theta) - \tfrac{1-\nu}{2}\sin 2\theta P_\sigma^1(\cos\theta),\\
\hat{h}_6(\theta) &= \sin^2\theta\ \omega P_\sigma(-\cos\theta) + (1-\nu)\cos\theta\sin\theta P_\sigma^1(-\cos\theta),\\
\hat{g}_2(\theta) &= \zeta\sin\theta P_\sigma^1(\cos\theta),\\
\hat{g}_6(\theta) &= \zeta\sin\theta P_\sigma^1(-\cos\theta),
\end{aligned} \tag{5.17}$$

according to (3.109).

In the Appendix one can find the proofs of the following properties of these functions

$$\begin{aligned}
&\hat{h}_2(0) = \hat{g}_2(0) = 0,\ \ \hat{h}_2(\theta) = 0(\theta^2),\ \ \hat{g}_2(\theta) = 0(\theta^2),\\
&\hat{h}_6(\theta) = 0(1) + 0(\theta^2 \ln\theta),\ \ \hat{g}_6(\theta) = 0(1) + 0(\theta^2\ln\theta),\\
&\hat{h}_6(0) = -\tfrac{2}{\pi}(1-\nu)\sin(\sigma\pi),\ \ \hat{g}_6(0) = -\tfrac{2}{\pi}\zeta\sin(\sigma\pi),
\end{aligned} \tag{5.18}$$

and their derivatives

$$\begin{array}{ll}
\text{(i)} & \frac{d\hat{h}_2}{d\theta} = 0(\theta),\\
\text{(ii)} & \frac{d\hat{g}_2}{d\theta} = 0(\theta),\\
\text{(iii)} & \frac{d\hat{h}_6}{d\theta} = 0(\theta^{-1}) + 0(\theta) + 0(\theta^2\ln\theta) + o(\theta),\\
\text{(iv)} & \frac{d\hat{g}_6}{d\theta} = 0(\theta^{-1}) + 0(\theta) + 0(\theta^3\ln\theta) + o(\theta).
\end{array} \tag{5.19}$$

The derivative of (5.16) is proportional to

$$\widetilde{K}(\theta) = [\widetilde{M}(\theta)]^{-2}[\widetilde{L}'(\theta)\widetilde{M}(\theta) - \widetilde{L}(\theta)\widetilde{M}'(\theta)] \tag{5.20}$$

with $(\ )' = d/d\theta$. Note first that

$$\widetilde{M}(0) = \alpha\overline{\beta}\widetilde{L}(0) + \overline{\alpha}\beta\overline{\widetilde{L}(0)} \tag{5.21}$$

with

$$\widetilde{L}(0) = iS,\ \ S = \frac{16\rho^2}{\pi^2}(1-\nu)|\sin(\sigma\pi)|^2. \tag{5.22}$$

Thus we have

$$\widetilde{M}(0) = -2dS,\ \ d \neq 0,\ \ S \neq 0, \tag{5.23}$$

with $d = Im(\overline{\alpha}\beta)$. The expression

$$F(\theta) = \widetilde{L}'(\theta)\widetilde{M}(\theta) - \widetilde{L}(\theta)\widetilde{M}'(\theta) \tag{5.24}$$

involves many singular terms due to the properties (5.19) (iii, iv). We shall prove below that all these singular terms cancel out and the rest of $F(\theta)$ tends to zero.

With the help of (5.19) we compute the singular terms of both the terms of $F(\theta)$:

$$\begin{aligned}
&\widetilde{L}'(\theta)\widetilde{M}(\theta) = -2dS(U - \overline{U}) + o(\theta),\\
&U = \hat{h}_6(\theta)\overline{\hat{g}_6'(\theta)} + \hat{h}_6'(\theta)\overline{\hat{g}_6(\theta)}
\end{aligned}$$

and

$$\widetilde{L}(\theta)\widetilde{M}'(\theta) = iS\{\alpha\overline{\beta}(U - \overline{U}) + \overline{\alpha}\beta(\overline{U} - U)\} + o(\theta).$$

Hence

$$\begin{aligned}\widetilde{L}(\theta)\widetilde{M}'(\theta) &= iS(\alpha\overline{\beta} - \overline{\alpha}\beta)(U - \overline{U}) + o(\theta)\\ &= 2d(i)^2 S(U - \overline{U}) + o(\theta),\end{aligned}$$

which confirms that $F(\theta)$ is nonsingular and vanish for $\theta \searrow 0$. The proof of (5.15) is finished.

5.3. This section is aimed at proving that

$$\Big|\frac{d^2\check{\mathcal{J}}(\mathbf{u}_{\theta_0})}{d(\theta_0)^2}(0)\Big| < +\infty. \tag{5.25}$$

By (5.20) this property is equivalent to the condition $|d\widetilde{K}/d\theta\ (\theta = 0)| < +\infty$. Let us compute

$$\widetilde{K}' = (\widetilde{M})^{-4}[\widetilde{L}''\widetilde{M}^3 - 2\widetilde{L}'\widetilde{M}'\widetilde{M}^2 + 2\widetilde{M}(\widetilde{M}')^2\widetilde{L} - \widetilde{M}^2\widetilde{M}''\widetilde{L}]. \tag{5.26}$$

Since $\widetilde{L}'(0)\widetilde{M}(0) = \widetilde{L}(0)\widetilde{M}'(0)$ we reduce the expression above to the form $\widetilde{K}'(\theta) = (\widetilde{M}(\theta))^{-2}F_1(\theta)$ with

$$F_1(\theta) = \widetilde{L}''(\theta)\widetilde{M}(\theta) - \widetilde{M}''(\theta)\widetilde{L}(\theta). \tag{5.27}$$

We shall prove that all singular terms of $F_1(\theta)$ cancel out. For small θ we have

$$F_1(\theta) = \widetilde{L}''(\theta)(-2dS) - \widetilde{M}''(\theta)(iS), \tag{5.28}$$

see (5.22), (5.23), and

$$\begin{aligned}\widetilde{L}''(\theta) = {}&\overline{(h_2'' - h_6'')}(g_2 + g_6) + 2\overline{(h_2' - h_6')}(g_2' + g_6') + \overline{(h_2 - h_6)}(g_2'' + g_6'')-\\ &- (h_2'' - h_6'')\overline{(g_2 + g_6)} - 2(h_2' - h_6')\overline{(g_2' + g_6')} - (h_2 - h_6)\overline{(g_2'' + g_6'')},\end{aligned} \tag{5.29}$$

$$\begin{aligned}\widetilde{M}''(\theta) = {}&\alpha\overline{\beta}[(g_6'' - g_2'')\overline{(h_2 - h_6)} + 2(g_6' - g_2')\overline{(h_2' - h_6')}+\\ &+ (g_6 - g_2)\overline{(h_2'' - h_6'')} + \overline{(g_6'' + g_2'')}(h_2 + h_6)+\\ &+ 2\overline{(g_6' + g_2')}(h_2' + h_6') + \overline{(g_6 + g_2)}(h_2'' + h_6'')]+\\ &+ \overline{\alpha}\beta[\overline{(g_6'' - g_2'')}(h_2 - h_6) + 2\overline{(g_6' - g_2')}(h_2' - h_6') + \overline{(g_6 - g_2)}(h_2'' - h_6'')+\\ &+ (g_6'' + g_2'')\overline{(h_2 + h_6)} + 2(g_6' + g_2')\overline{(h_2' + h_6')} + (g_6 + g_2)\overline{(h_2'' + h_6'')}]-\\ &- 2\alpha\beta\overline{[g_2''h_6 + 2g_2'h_6' + g_2h_6'' + g_6''h_2 + 2g_6'h_2' + g_6h_2'']}-\\ &- 2\overline{\alpha\beta}[g_2''h_6 + 2g_2'h_6' + g_2h_6'' + g_6''h_2 + 2g_6'h_2' + g_6h_2''],\end{aligned} \tag{5.30}$$

the hat "$\wedge$" over g_i and h_i being omitted for brevity. This sign will be omitted up to the end of this section.

In the vicinity of $\theta = 0$ the functions g_6'', h_6'', g_2'', h_2'' behave as follows

$$\begin{array}{ll} h_2''(\theta) = 0(\theta^{-1}), & g_2''(\theta) = 0(\theta^{-1}), \\ h_6''(\theta) = 0(\theta^{-2}), & g_6''(\theta) = 0(\theta^{-2}). \end{array} \tag{5.31}$$

The properties (5.18), (5.19) and (5.31) determine the singular behaviour of $F_1(\theta)$ at $\theta = 0$. One can group the terms of (5.28) into three functions as follows

$$F_1(\theta) = F_1^{(a)}(\theta) + F_1^{(b)}(\theta) + F_1^{(c)}(\theta), \tag{5.32}$$

where

$$\begin{array}{l} F_1^{(a)}(\theta) = 2\alpha\beta(\overline{g_2''h_6} + \overline{g_6h_2''}) + 2\overline{\alpha\beta}(g_2''h_6 + g_6h_2''), \\ F_1^{(b)}(\theta) = -2dS[(\overline{h_2''}g_6 + h_6\overline{g_2''}) - (\overline{h_6}g_2'' + h_2''\overline{g_2})] - \\ -\, iS(\alpha\overline{\beta} + \overline{\alpha}\beta)(g_2''\overline{h_6} + \overline{g_6}h_2'' + g_6\overline{h_2''} + \overline{g_2''}h_6), \\ F_1^{(c)}(\theta) = -2dS[-\overline{h_6''}g_6 - 2\overline{h_6'}g_6' - \overline{h_6}g_6'' + h_6''\overline{g_6} + 2h_6'\overline{g_6'} + \overline{g_6''}h_6] \\ -\, iS\{\alpha\overline{\beta}[-g_6''\overline{h_6} - 2g_6'\overline{h_6'} - g_6\overline{h_6''} + \overline{g_6''}h_6 + 2\overline{g_6'}h_6' + \overline{g_6}h_6''] + \\ +\, \overline{\alpha}\beta[-\overline{g_6''}h_6 - 2\overline{g_6'}h_6' - \overline{g_6}h_6'' + g_6''\overline{h_6} + 2g_6'\overline{h_6'} + g_6\overline{h_6''}]\}. \end{array} \tag{5.33}$$

Note that all terms of $F_1^{(a)}$ and $F_1^{(b)}$ are of order $0(\theta^{-1})$ and the terms of $F_1^{(c)}$ are of order $0(\theta^{-2})$. To prove that all singular terms of $F_1^{(a)}$ cancel out it is sufficient to show that the function

$$F_2(\theta) = g_2''(\theta)h_6(\theta) + g_6(\theta)h_2''(\theta) \tag{5.34}$$

is non-singular at $\theta = 0$.

The singular term of g_2'' has the form

$$g_2'' \sim 2\zeta\frac{\cos^2\theta}{\sin\theta}P_\sigma^1(\cos\theta)$$

and the non-vanishing for $\theta = 0$ term of h_6 reads

$$h_6 \sim (1-\nu)\cos\theta\sin\theta P_\sigma^1(-\cos\theta).$$

The singular term of h_2'' reads

$$h_2'' \sim -\frac{(1-\nu)}{2}\frac{(6\cos^2\theta - 2)\cos\theta}{\sin\theta}P_\sigma^1(\cos\theta),$$

while g_6 equal to

$$g_6 = \zeta \sin\theta P^1_\sigma(-\cos\theta)$$

is non-zero in $\theta = 0$. Thus $F_2(\theta)$ and $F_1^{(a)}(\theta)$ are finite at $\theta = 0$.
Let us proceed now to proving that $|F_1^{(b)}(0)| < +\infty$. Let

$$A(\theta) = \overline{g_2''(\theta)} h_6(\theta) + \overline{h_2''(\theta)} g_6(\theta). \tag{5.35}$$

Then

$$F_1^{(b)}(\theta) = -2dS(A - \overline{A}) - iS(\alpha\overline{\beta} + \overline{\alpha}\beta)(A + \overline{A}).$$

Since $\alpha\overline{\beta} + \overline{\alpha}\beta = 2c$, we have

$$F_1^{(b)}(\theta) = -2dS(A - \overline{A}) - iS.2c(A + \overline{A}),$$

or

$$F_1^{(b)}(\theta) = -2S[(d + ic)A + (ic - d)\overline{A}].$$

One can prove that $|A(0)| < +\infty$; this proof is similar to that concerning the function $F_2(\theta)$. Hence $F_1^{(b)}$ is non-singular at $\theta = 0$.
Let us prove that $|F_1^{(c)}(0)| < +\infty$. Let us define

$$B(\theta) = h_6''(\theta)\overline{g_6(\theta)} + 2h_6'(\theta)\overline{g_6'(\theta)} + h_6(\theta)\overline{g_6''(\theta)}. \tag{5.36}$$

Note that

$$F_1^{(c)}(\theta) = -2dS(B - \overline{B}) - iS[\alpha\overline{\beta}(B - \overline{B}) + \overline{\alpha}\beta(\overline{B} - B)]$$

but $\alpha\overline{\beta} = c + id$, which gives $F_1^{(c)}(\theta) = 0$. Thus the property (5.25) holds good.

Appendix

The aim of this appendix is to report the basic properties of the functions $P^m_\sigma(x)$, $R^m_\sigma(\theta)$ and $S^m_\sigma(\theta)$ frequently used in the body of the present work.

Let us recall the Hobson formula(see Wang and Guo (1989), p. 256)

$$P^m_\sigma(x) = (-1)^m \frac{1}{m!}\Big(\frac{1-x}{1+x}\Big)^{\frac{m}{2}} \frac{\Gamma(\sigma + m + 1)}{\Gamma(\sigma - m + 1)} F\Big(-\sigma, \sigma+1, 1+m, \frac{1-x}{2}\Big) \tag{A.1}$$

for $|x| < 1$; Γ represents the gamma function and F is the hypergeometric function. By using the relation

$$P_\sigma^{-m}(x) = (-1)^m \frac{\Gamma(\sigma - m + 1)}{\Gamma(\sigma + m + 1)} P_\sigma^m(x) \tag{A.2}$$

we find the equation valid for $|x| < 1$

$$P_\sigma^{-m}(x) = \frac{1}{m!}\Big(\frac{1-x}{1+x}\Big)^{\frac{m}{2}} F\Big(-\sigma, \sigma+1, 1+m, \frac{1-x}{2}\Big), \tag{A.3}$$

used by Niordson (1985, Eq. (13.60)).
If we put $x = \cos\theta$ the functions $P_\sigma^{-m}(\cos\theta)$ become singular for $\theta \to \pi$ (then $x \to -1$). Let us recall how these functions behave at $x = -1$. After Wang and Guo (1989, p. 257) we report the formula

$$\begin{aligned} &P_\sigma^m(x) = -\tfrac{\sin\sigma\pi}{\pi}(m-1)!\Big(\tfrac{1-x}{1+x}\Big)^{\frac{m}{2}} \textstyle\sum_{k=0}^{m-1} \tfrac{(-\sigma)_k(\sigma+1)_k}{k!(1-m)_k}\Big(\tfrac{1+x}{2}\Big)^k + \\ &+ \tfrac{\sin\sigma\pi}{\pi}\tfrac{\Gamma(\sigma+m+1)}{2^m\Gamma(\sigma-m+1)}(1-x^2)^{\frac{m}{2}} \textstyle\sum_{k=0}^{\infty} \tfrac{(-\sigma+m)_k(\sigma+1+m)_k}{k!(k+m)!}. \\ &.\Big(\tfrac{1+x}{2}\Big)^k\Big\{\psi(-\sigma+m+k) + \psi(\sigma+1+m+k) - \psi(1+m+k) - \\ &- \psi(1+k) + \ln\tfrac{1+x}{2}\Big\} \end{aligned} \tag{A.4}$$

valid for $m = 0, 1, 2, \ldots$ and $|x| < 1$. Here

$$\begin{aligned} &(z)_n = z(z+1)\ldots(z+n-1) \ \text{ for } \ n \geq 1 \\ &(z)_0 = 1 \end{aligned} \tag{A.5}$$

and $\psi(.)$ represents the logarithmic derivative of Γ. For $m = 0$ the finite sum above does not appear.
Consider the case of $m = 1$. Then

$$\begin{aligned} &P_\sigma^1(x) = \tfrac{-\sin\sigma\pi}{\pi}\Big(\tfrac{1-x}{1+x}\Big)^{1/2} + \tfrac{\sin\sigma\pi}{\pi}\tfrac{\Gamma(\sigma+2)}{2\Gamma(\sigma)}\sqrt{1-x^2}\textstyle\sum_{k=0}^{\infty} \\ &\tfrac{(1-\sigma)_k(2+\sigma)_k}{k!(k+1)!}\Big(\tfrac{1+x}{2}\Big)^k.\Big\{\psi(-\sigma+1+k) + \psi(\sigma+2+k) - \\ &- \psi(2+k) - \psi(1+k) + \ln\tfrac{1+x}{2}\Big\}. \end{aligned} \tag{A.6}$$

One can find now that

$$\lim_{x\to -1}(1-x^2)^{1/2}P_\sigma^1(x) = -\frac{2\sin\sigma\pi}{\pi}, \tag{A.7}$$

or, see (3.66)

$$\lim_{\theta\to 0} \sin\theta P^1_\sigma(-\cos\theta) = \lim_{\theta\to 0} [\sin\theta(R^{-1}_\sigma(\pi-\theta)+iS^{-1}_\sigma(\pi-\theta))] = -\frac{2\sin\sigma\pi}{\pi}. \quad (A.8)$$

Since by (3.60) $\sigma = a + ib$, we find

$$\sin(\sigma\pi) = \sin(\pi a)\cosh(\pi b) + i\cos(\pi a)\sinh(\pi b) \quad (A.9)$$

and

$$\begin{array}{l} \lim_{\theta\to 0}(\sin\theta R^{-1}_\sigma(\pi-\theta)) = -\frac{2}{\pi}\sin(\pi a)\cosh(\pi b), \\ \lim_{\theta\to 0}(\sin\theta S^{-1}_\sigma(\pi-\theta)) = -\frac{2}{\pi}\cos(\pi a)\sinh(\pi b). \end{array} \quad (A.10)$$

Hence we find the results (5.18) and (5.9).

Let us proceed now to proving the results (5.19). The known formula(see Niordson (1985), p. 306)

$$(1-x^2)\frac{dP^m_\sigma}{dx} = -(1-x^2)^{1/2}P^{m+1}_\sigma - mxP^m_\sigma \quad (A.11)$$

implies

$$\frac{dP^m_\sigma(\cos\theta)}{d\theta} = P^{m+1}_\sigma(\cos\theta) + m\operatorname{ctg}\theta P^m_\sigma(\cos\theta), \quad (A.12)$$

since $dx = -(1-x^2)^{1/2}d\theta$, if $x = \cos\theta$.

Substitution $x = -\cos\theta$ or $\theta \Rightarrow \pi - \theta$ gives

$$\frac{dP^m_\sigma(-\cos\theta)}{d\theta} = P^{m+1}_\sigma(-\cos\theta) - m\operatorname{ctg}\theta P^m_\sigma(-\cos\theta). \quad (A.13)$$

Now we can express the function $\hat{g}_6'(\theta)$, see (5.17), as follows

$$\frac{d\hat{g}_6}{d\theta} = \zeta\sin\theta P^2_\sigma(-\cos\theta). \quad (A.14)$$

The representation (A.4) gives us the information on the behaviour of this function for small θ. This leads us to the property (iv) of (5.19). The proof of (iii) of (5.19) is similar.

Similarly one finds

$$\frac{d\hat{g}_2}{d\theta} = \zeta[2\cos\theta P^1_\sigma(\cos\theta) + \sin\theta P^2_\sigma(\cos\theta)]. \quad (A.15)$$

Taking into account that $P_\sigma^m(\cos\theta) \to 0$ if $\theta \searrow 0$ for $m = 1, 2$ (see Wang and Guo (1989), p. 256) we confirm (ii) of (5.19). Note that

$$\frac{d}{d\theta}(\sin^2\theta P_\sigma(\cos\theta)) = \omega[\sin 2\theta P_\sigma(\cos\theta) + \sin^2\theta P_\sigma^1(\cos\theta)] \tag{A.16}$$

since $dP_\sigma(\cos\theta)/d\theta = P_\sigma^1(\cos\theta)$. Taking into account that $\lim P_\sigma(\cos\theta) = 1$ if $\theta \searrow 0$ we conclude that the expression (A.16) vanishes if $\theta \searrow 0$. Compute now

$$\frac{d}{d\theta}(\sin 2\theta P_\sigma^1(\cos\theta)) = 2\cos 2\theta P_\sigma^1(\cos\theta) + \sin 2\theta[P_\sigma^2(\cos\theta) + \operatorname{ctg}\theta P_\sigma^1(\cos\theta)]. \tag{A.17}$$

By the same arguments this expression tends to zero if $\theta \searrow 0$. By (A.16) and (A.17) the formula (i) of (5.19) follows.

Let us recall now the formulae which have made it possible to derive Eqs. (3.72)-(3.74), (3.79), (3.84) and (3.88).

The formulae (3.66) and (A.13) (valid for negative m's) imply

$$\frac{d}{d\theta}R_\sigma^m = R_\sigma^{m-1} - m\operatorname{ctg}\theta R_\sigma^m, \tag{A.18}$$

$$\frac{d}{d\theta}S_\sigma^m = S_\sigma^{m-1} - m\operatorname{ctg}\theta S_\sigma^m. \tag{A.19}$$

Hence we find

$$\begin{aligned} \frac{d^2R_\sigma^m}{d\theta^2} &= \frac{m^2\cos^2\theta+m}{\sin^2\theta}R_\sigma^m + (1-2m)\operatorname{ctg}\theta R_\sigma^{m-1} + R_\sigma^{m-2}, \\ \frac{d^2S_\sigma^m}{d\theta^2} &= \frac{m^2\cos^2\theta+m}{\sin^2\theta}S_\sigma^m + (1-2m)\operatorname{ctg}\theta S_\sigma^{m-1} + S_\sigma^{m-2}. \end{aligned} \tag{A.20}$$

Moreover

$$\begin{aligned} R_\sigma^{m-2}(\theta) &= 2(m-1)\operatorname{ctg}\theta R_\sigma^{m-1}(\theta) - (1-m^2+m)R_\sigma^m + 2\rho^2 S_\sigma^m(\theta), \\ S_\sigma^{m-2}(\theta) &= 2(m-1)\operatorname{ctg}\theta S_\sigma^{m-1}(\theta) - (1-m^2+m)S_\sigma^m - 2\rho^2 R_\sigma^m(\theta) \end{aligned} \tag{A.21}$$

since

$$\Big(\Delta + \frac{1+i2\rho^2}{R^2}\Big)(R_\sigma^m + iS_\sigma^m)\cos m\phi = 0. \tag{A.22}$$

Let us report now the power expansions of $R_\sigma^m(\theta)$ and $S_\sigma^m(\theta)$. Let us recall Eqs. (13.66) of Niordson (1985) :

$$\begin{aligned} R_\sigma^m(\theta) &= \tfrac{1}{m!}\Big(\operatorname{tg}\tfrac{\theta}{2}\Big)^m\Big[1+\textstyle\sum_{n=0}^\infty \tilde{a}_n(x)\Big], \quad x=\cos\theta, \\ S_\sigma^m(\theta) &= \tfrac{1}{m!}\Big(\operatorname{tg}\tfrac{\theta}{2}\Big)^m \textstyle\sum_{n=0}^\infty \tilde{b}_n(x) \end{aligned} \tag{A.23}$$

where the sequences $\{\tilde{a}_n\}$, $\{\tilde{b}_n\}$ are defined reccursively :

$$\tilde{a}_0(x) = -\frac{\xi(1-x)}{2(1+m)}, \quad \tilde{b}_0(x) = -\frac{\eta}{2}\frac{1-x}{m+1} \tag{A.24}$$

where $\sigma(\sigma+1) = \xi + i\eta$, hence $\xi = 1$ and $\eta = 2\rho^2$ in our case ; but the parameters ξ, η will be used later ; for $n \geq 1$:

$$\begin{aligned} \tilde{a}_n(x) &= \tfrac{[(n^2+n-\xi)\tilde{a}_{n-1}(x)+\eta\tilde{b}_{n-1}(x)]}{2(n+1)(n+m+1)}(1-x), \\ \tilde{b}_n(x) &= \tfrac{[(n^2+n-\xi)\tilde{b}_{n-1}(x)-\eta\tilde{a}_{n-1}(x)](1-x)}{2(n+1)(n+m+1)}. \end{aligned} \tag{A.25}$$

Let us rewrite (A.23) in the form

$$\begin{aligned} R_\sigma^m(\theta) &= \tfrac{1}{m!}\Big(\operatorname{tg}\tfrac{\theta}{2}\Big)^m\Bigg(1+\textstyle\sum_{n=0}^\infty a_n\Big(\tfrac{1-\cos\theta}{2}\Big)^n\Bigg), \\ S_\sigma^m(\theta) &= \tfrac{1}{m!}\Big(\operatorname{tg}\tfrac{\theta}{2}\Big)^m \textstyle\sum_{n=0}^\infty b_n\Big(\tfrac{1-\cos\theta}{2}\Big)^n, \end{aligned} \tag{A.26}$$

where $\{a_n\}$, $\{b_n\}$ are defined reccursively. We start with

$$a_0 = -\frac{\xi}{m+1}, \quad b_0 = \frac{-\eta}{m+1}, \quad \mathbf{X}_0 = \begin{bmatrix} a_0 \\ b_0 \end{bmatrix}, \tag{A.27}$$

define the matrices

$$\mathbf{X}_n = \begin{bmatrix} a_n \\ b_n \end{bmatrix}, \quad \mathbf{a}_n = \begin{bmatrix} c_n & \eta \\ -\eta & c_n \end{bmatrix} \tag{A.28}$$

$$c_n = n^2 + n - \xi, \quad d_n = (n+1)(n+m+1)$$

and we obtain the recurrence formula

$$\mathbf{X}_n = \frac{1}{d_n}\mathbf{a}_n\mathbf{X}_{n-1}. \tag{A.29}$$

Its solution reads

$$\mathbf{X}_n = \left(\prod_{k=1}^{n} (d_k)^{-1} \right) \underbrace{(\mathbf{a}_n \mathbf{a}_{n-1} \ldots \mathbf{a}_1)}_{\mathcal{A}_n} \mathbf{X}_0. \tag{A.30}$$

The matrix $\mathcal{A}_n$ can be found explicitly.
To this end we decompose the matrix $\mathbf{a}_k$ as below

$$\mathbf{a}_k = (c_k + i\eta)\mathbf{S}_1 + (c_k - i\eta)\mathbf{S}_2 \tag{A.31}$$

with

$$\mathbf{S}_1 = \frac{1}{2} \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}, \;\; \mathbf{S}_2 = \frac{1}{2} \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}. \tag{A.32}$$

Note that $\mathbf{S}_\alpha$ are projections of usual properties

$$\mathbf{S}_1\mathbf{S}_1 = \mathbf{S}_1, \;\; \mathbf{S}_2\mathbf{S}_2 = \mathbf{S}_2, \;\; \mathbf{S}_1\mathbf{S}_2 = \mathbf{S}_2\mathbf{S}_1 = \mathbf{0}. \tag{A.33}$$

Hence

$$\underline{\mathcal{A}}_n = \left(\prod_{k=1}^{n} (c_k + i\eta) \right) \mathbf{S}_1 + \left(\prod_{k=1}^{n} (c_k - i\eta) \right) \mathbf{S}_2 \tag{A.34}$$

and we find eventually

$$a_n = -\frac{1}{m+1} Re[t_n(\xi - i\eta)], \;\; b_n = -\frac{1}{m+1} Re[t_n(\eta + i\xi)] \;, \tag{A.35}$$

where

$$t_0 = 1, \;\; t_n = \prod_{k=1}^{n} \frac{(k^2 + k - \xi + i\eta)}{(k+1)(k+1+m)}. \tag{A.36}$$

Algorithm for finding $w_{\theta_0}(\pi/2)$
Given : ν , R/h.
Find :

$$\rho^2 = [3(1-\nu^2)]^{1/2} \tfrac{R}{h},$$
$$\zeta = 1 - i2\rho^2, \;\; \omega = \nu - i2\rho^2,$$
$$\alpha = \zeta P_\sigma^1(0), \quad \beta = \omega P_\sigma(0),$$

where $\sigma(\sigma+1) = 1 + i2\rho^2$.

Define

$$
\begin{aligned}
&h_2(\theta) = \omega P_\sigma(\cos\theta) - (1-\nu)\operatorname{ctg}\theta P_\sigma^1(\cos\theta),\\
&h_6(\theta) = \omega P_\sigma(-\cos\theta) + (1-\nu)\operatorname{ctg}\theta P_\sigma^1(-\cos\theta),\\
&g_2(\theta) = \zeta P_\sigma^1(\cos\theta),\\
&g_6(\theta) = \zeta P_\sigma^1(-\cos\theta),\\
&k(\theta) = h_2(\theta) - h_6(\theta), \quad l(\theta) = h_2(\theta) + h_6(\theta),\\
&m(\theta) = g_2(\theta) - g_6(\theta), \quad n(\theta) = g_2(\theta) + g_6(\theta),\\
&L(\theta) = Im\{\overline{k(\theta)}n(\theta)\},\\
&M(\theta) = Re\{\alpha\overline{\beta}[\overline{n(\theta)}l(\theta) - m(\theta)\overline{k(\theta)}] - 2\alpha\beta(\overline{g_2(\theta)h_6(\theta)} + \overline{g_6(\theta)h_2(\theta)}]\},\\
&\overline{w}_{\theta_0}(\pi/2) = 4\rho^2|P_\sigma(0)|^2\tfrac{L(\theta_0)}{M(\theta_0)}\ ;\ \text{(non-dimensional result)},\\
&w_{\theta_0}(\pi/2) = \tfrac{PR^3}{D}\overline{w}_{\theta_0}(\pi/2)\ \ ;\ [w_{\theta_0}] = m
\end{aligned}
$$

where

$$
D = \frac{Eh^3}{12(1-\nu^2)}.
$$

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